Tilings of Modified Rectangles by Height-1 Ribbon Tile Pentominoes

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Abstract

Which dimensions of a and b satisfy tilings of modified rectangles $M(a, b)$ by ribbon tile pentominoes? We will answer this question using tile invariants developed in prior research on the mathematics of tiling, as well as the use of inductive lemmas. The set of tiles we use for this study are height-1 ribbon tile pentominoes, which we later define. Modified rectangles are $[a \times b]$ rectangles with height a and width b, and the additional feature that the top left and bottom right square tiles are removed.

1 Introduction to Tiling

The mathematics of tiling has been a deeply explored topic amongst mathematicians for several years. In simplistic terms, one can think of tiling as tetris for mathematics. Rather than tiling rows and columns in a video game, one is tiling a geometric region. When exploring tilings, we first need two things-a family of regions to tile, and set of tiles to tile them with. For a region to be tiled, the set of tiles must completely cover the region with no overlap or holes. We ask ourselves whether or not a region can be tiled by a given set. If a region is tiled, we can prove this by simply drawing the tiling of that region. If a region cannot be tiled, however, the proof behind this is a much more deeply investigated question. Mathematicians Igor Pak and Michael Hitchman discuss properties of both tiled and untiled regions, as well as introduce useful tools we can use to prove the nonexistence of tilings.

Igor Pak proves that rectangles are tiled by the set of tiles below if and only if their area is divisible by 10. We later prove a similar property using the same tile set with a different family of regions.

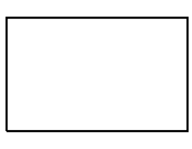


Figure 1, Rectangle

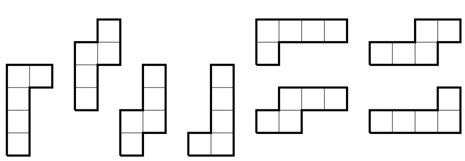


Figure 2, Tile Set

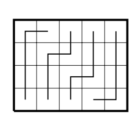


Figure 3, Tiled $[4 \times 5]$ Rectangle

Michael Hitchman proves which regions in Figure 4 are tiled by the set in Figure 5. We later work with this same family of regions, and discuss Hitchman's results and the parallels between his approach to answering this question and ours.

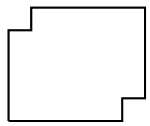


Figure 4, A Modified rectangle

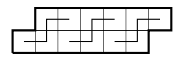


Figure 6, A Tiled Modified Rectangle

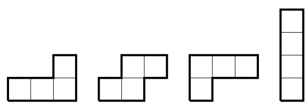


Figure 5, Tile Set

The key to Pak and Hitchman's arguments is their use of tile invariants. Tile invariants are tools that we can use to rule out regions that cannot tiled by a given set. We also use tile invariants to answer the question: Which modified rectangles are tiled by height-1 ribbon tile pentominoes? We define these and other necessary terms in the next section to answer our question.

2 Tile Invariants

Definition 1. *A* ***ribbon tile*** *of area* m *is a polyomino (a region consisting of 2 or more squares connected from edge to edge) consisting of m squares laid out in a path, such that from an initial square, each step either goes up or to the right. A zero denotes a step to the right, and a 1 denotes a step up.*

Definition 2. *A* ***height-0 ribbon tile*** *is a ribbon tile whose binary code has a sum that is congruent to 0* $\pmod 2$*. We denote these tiles* T_n*.*

Example. Tile 1010 is a height-0 ribbon tile since its binary code has an even sum.

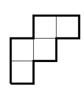


Figure 7

Definition 3. *A* ***height-1 ribbon tile*** *is a ribbon tile whose binary code has a sum that is congruent to 1* $\pmod 2$*. We denote these tiles* T'_n*.*

Example. Tile 1101 is a height-1 ribbon tile since its binary code has an odd sum.

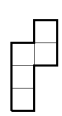


Figure 8

Definition 4. ***Height-1 ribbon tile pentominoes***, *denoted T_5', have area 5 and are defined as the set {0001, 0010, 0100, 1000, 0111, 1011, 1101, 1110}.*

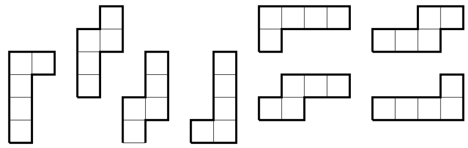


Figure 9, T_5'

Definition 5. *A* ***modified rectangle*** *is an $[a \times b]$ rectangle with the top left and bottom right squares removed, denoted $M(a,b)$.*

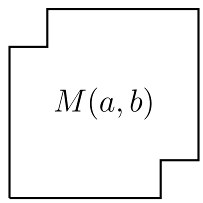


Figure 10

Now that we have defined our family of regions and set of tiles, we can define a tile invariant.

Definition 6. *Let $T = \{t_1, t_2, ..., t_n\}$ have n tiles each with area m, and let $\mathbf{R}$ be a family of regions. Suppose α is a tiling of a region $R \in \mathbf{R}$. Let $a_i(\alpha)$ equal the number of times tile t_i appears in the tiling α. Then a* ***tile invariant*** *is the sum:*

$$k_1 \cdot a_1(\alpha) + k_2 \cdot a_2(\alpha) + ... + k_n \cdot a_n(\alpha) = c(R) \tag{1}$$

where c(R) is constant.

In other words, a tile invariant is a constant of sum of tiles used in any tiling of a given region. Invariants are extremely useful tools when proving the nonexistence of tilings. We define three invariants below.

Definition 7. *The* ***area of a region*** *R is denoted $|R|$. This value is equal to the total number of square units in that region.*

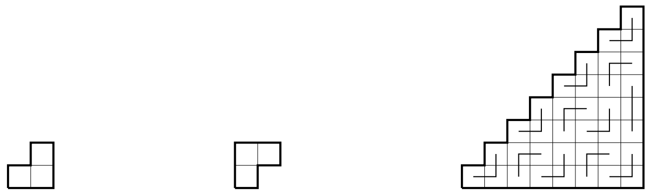
Definition 8. *Suppose the area of each of the tiles in $T = \{t_1, ..., t_n\}$ is m. Then for each region R in $\mathbf{R}$, if α is a tiling of R,*

$$a_1(\alpha) + a_2(\alpha) + ... + a_n(\alpha) = \frac{|R|}{m}. \tag{2}$$

Example. We can use the area invariant to prove that $M(3,5)$ is not tiled by T_5'. Let $R = M(3,5)$. Then $|R| = 13$. By our area invariant, in order for this region to be tiled, its area must be divisible by the 5. 13 is not divisible by 5, therefore this region is not tiled. We later use the area invariant to prove which $M(a,b)s$ are tiled by T_5'.

Definition 9. ***The Conway/Lagarias Invariant.*** *In any tiling of a staircase region S_n by the set T_3, the number of t_2 tiles used minus the number of t_3 tiles used is constant.*

Hitchman provides a useful example of this invariant in application with the staircase region S_8:



t_2 t_3 S_8

Figure 11

We can use our Conway/Lagarias invariant to show $\{t_2, t_3\}$ does not tile S_8. By our area invariant, $a_2 + a_3 = 12$, and by our Conway/Lagarias invariant $a_2 - a_3 = 3$. There are no integer solutions to this system, so $\{t_2, t_3\}$ does not tile S_8. The Conway/Lagarias invariant is ultimately what inspired the discovery of other invariants used in tiling. One of these invariants described below was discoverd by Pak, and provides a useful tool to prove the nonexistence of tilings.

Definition 10. *If R is a simply connected region tileable by T_n, then the total number of height-1 tiles used in any tiling of R is constant modulo 2. We call this the* ***height invariant****, denoted $h(R)$.*

Hitchman proves the following useful test to show the nonexistence of tilings by using Pak's height invariant:

Lemma 1. ***Tileability test for T'_n.*** *Suppose there exists a tiling of a simply connected region R by T_n in which an odd number of height-0 tiles are used. Then the set T'_n of height-1 tiles does not tile R.*

Proof. Suppose T_n tiles R with $a+b$ tiles where a counts the number of height-1 tiles and b counts the number of height-0 tiles, and suppose $b > 1$ is odd. Then $h(R) = a \cdot 1 + b \cdot 0 \equiv a \pmod 2$. If T'_n tiles R then it does so with $a + b$ tiles, in which case we would also have $h(R)$ ≡a+b $\pmod 2$. But these two descriptions of $h(R)$ would imply $b \equiv 0 \pmod 2$, a contradiction since b is odd. Thus, no tiling of R by height-1 tiles exists. $\square$

Example. Let us convince ourselves that our height invariant does indeed work when proving the nonexistence of tilings. Consider the given tiling below.

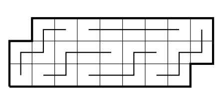


Figure 12, $M(3, 9)$

This region is tiled by three height-0 tiles and two height-1 tiles in T_5. Then $h(R) \equiv 0 \pmod 2$. If we assume that T_5' also tiles this region, then it must do so with five tiles as well by our area invariant. Then $h(R) \equiv 1 \pmod 2$, a contradiction since $h(R)$ is even in our first case and odd in our second case. Thus T_5' does not tile $M(3,9)$.

We use our tileability test to prove the nonexistence of tilings by T_5'.

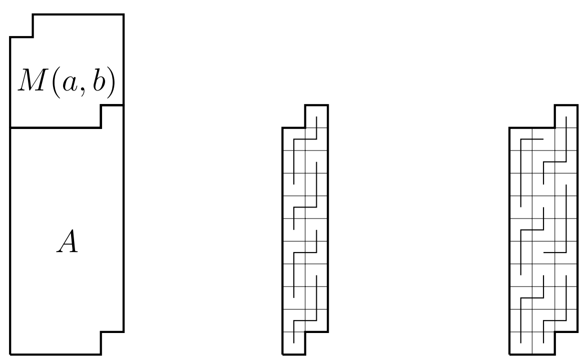
3 Main Results

Theorem 1. *The set T_5' tiles $M(a,b)$ if and only if $ab \equiv 2 \pmod{10}$.*

Before we go through the proof of this theorem, we introduce the following useful lemmas.

Lemma 2. *If $M(a,b)$ is tiled by T_5' then $M(a+10,b)$ and $M(a,b+10)$ are tiled by T_5'.*

Proof. Suppose $M(a,b)$ is tiled by T_5'. Consider $M(a+10,b)$. $M(a,b)$ is the top portion of this figure. Let A be the bottom portion of $M(a+10,b)$. We show that A is tiled for all b >1:



b=2; $M(11,2)$ b=3

For all $b \geq 4$, A can be partitioned into the following tiled regions:

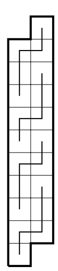


Figure 12, $M(11,2)$

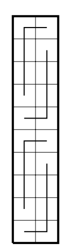


Figure 13, $[10 \times 2]$ Rectangle

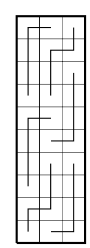
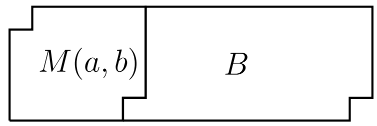
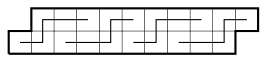


Figure 14, $[10 \times 3]$ Rectangle

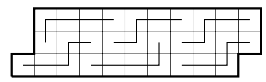
Therefore, for all even $b \geq 4$, A can be partitioned into one tiled region $M(11,2)$ and $\frac{b-2}{2}$ $[10 \times 2]$ tiled rectangles. For all odd $b \geq 5$, A can be partitioned into one tiled region $M(11,2)$, $\frac{b-5}{2}$ $[10 \times 2]$ tiled rectangles, and one $[10 \times 3]$ tiled rectangle. Thus A is tiled by T_5' for all $b > 1$ and $M(a+10,b)$ is tiled by T_5'.

Consider $M(a, b+10)$. $M(a,b)$ Is the left portion of this region. Let B be the right portion of this region. We show that B is tiled for all $a > 1$:





a=2; $M(2,11)$



a=3

Notice that for all $a \geq 4$, B can be partitioned into the following tiled regions:

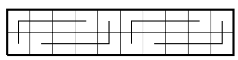


FIgure 15, $[2 \times 10]$

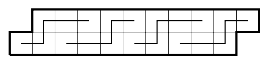


Figure 16, $M(2,11)$

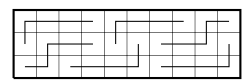


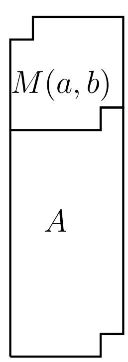
Figure 17, $[3 \times 10]$

Therefore, for all even $a \geq 4$, B can be partitioned into one tiled $M(2,11)$ and $\frac{a-2}{2}$ $[2 \times 10]$ tiled rectangles, and for all odd $a \geq 5$, B can be partitioned into one tiled region $M(2,11)$, one tiled $[3 \times 10]$ rectangle, and $\frac{a-5}{2}$ tiled $[2 \times 10]$ rectangles. Thus, B is tiled by T_5' for all $a > 1$ and $M(a, b+10)$ is tiled by T_5'.

$\square$

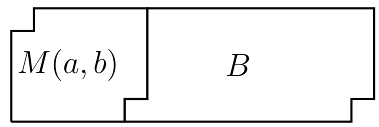
Lemma 3. *If an odd number of height-0 tiles in T_5 tiles $M(a,b)$ then an odd number of height-0 tiles in T_5 tiles $M(a+10,b)$ and $M(a,b+10)$ also.*

Proof. Suppose $M(a,b)$ is tiled by an odd number of height-0 tiles in T_5. Consider $M(a+10,b)$. $M(a,b)$ is the top portion of this region. Let A be the bottom portion of $M(a+10,b)$. We show A is tiled by an even number of height-0 tiles in T_5' for all $b > 1$:



By our proof of Lemma 2, A is tiled by zero height-0 tiles in T_5 for all b>1. Therefore $M(a+10,b)$ is tiled by an odd number of height-0 tiles in T_5.

Consider $M(a, b+10)$. $M(a, b)$ Is the left portion of this region. Let B be the right portion of $M(a, b+10)$. We show B is tiled by an even number of height-0 tiles in T_5' for all a >1:



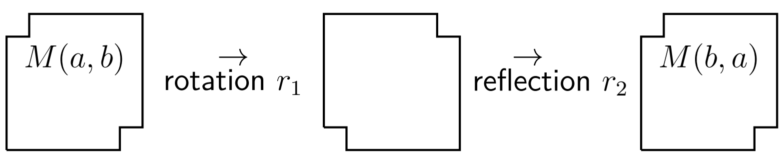
By our proof of Lemma 2, B is tiled by zero height-0 tiles in T_5 for all a>1. Therefore $M(a, b+10)$ is tiled by an odd number of height-0 tiles in T_5.

□

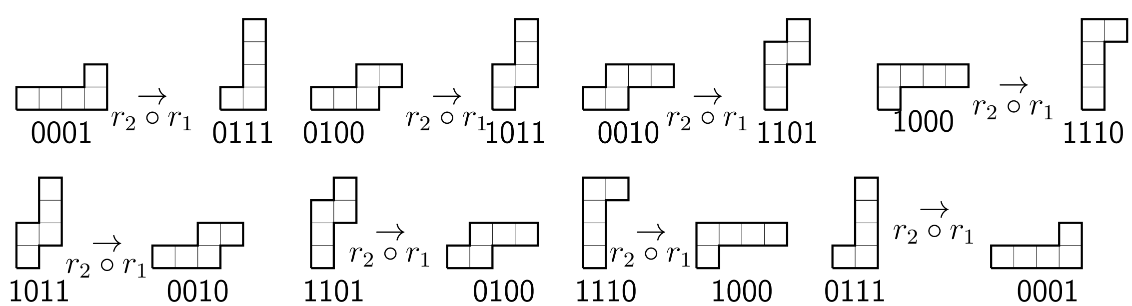
Lemma 4. ***Symmetry Lemma.***

1. *The set T_5' tiles $M(a,b)$ if and only if T_5' tiles $M(b,a)$.*
2. *An odd number of height-0 tiles in T_5 tiles $M(a,b)$ if and only if an odd number of height-0 tiles in T_5 also tiles $M(b,a)$.*

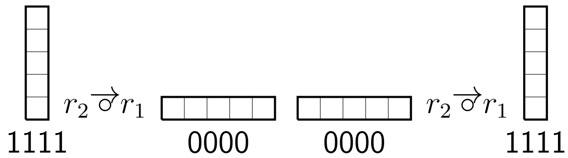
Proof. Suppose T_5' tiles $M(a,b)$. If we rotate $M(a,b)$ 90 degrees clockwise and reflect it about its right side, we have $M(b,a)$:



Notice that all tiles in T_5' still remain in the simply connected region $M(a,b)$ after applying both the rotation clockwise 90 degrees and a reflection about their right side. Therefore T_5' also tiles $M(b,a)$:



Suppose an odd number of height-0 tiles in T_5 tiles $M(a,b)$. We focus specifically on the two height-0 tiles we use in our proof: {1111, 0000}. If we rotate $M(a,b)$ and reflect it about its right side, 1111 and 0000 still remain in $M(b,a)$. Thus $M(b,a)$ is tiled by an odd number of height-0 tiles in T_5.



□

Now we use these Lemmas to prove our **Tileability Theorem**: The set T_5' tiles $M(a,b)$ if and only if $ab \equiv 2 \pmod{10}$.

Proof. Suppose T_5' tiles $M(a,b)$. Observe possible values of a and b such that $a, b > 1$ below.

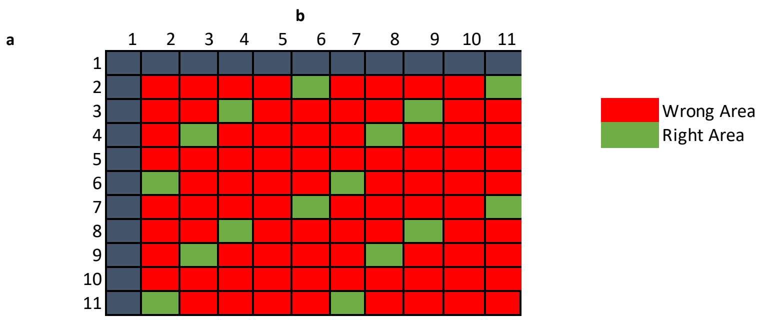
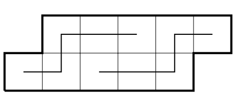


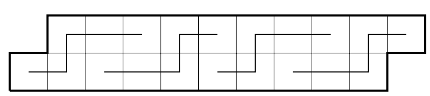
Figure 1: Base Case $M(a,b)$s

By our area invariant, the area of $M(a,b)$ must be divisible by 5. Thus we can rule out all base case regions with area not divisible by 5. Dimensions of modified rectangles with the right area are shaded in green and regions with the wrong area are in red. Of our 16 possible regions, 12 of them are tiled. We show 6 of these tiled regions below, and by our Symmetry Lemma the other 6 are also tiled:

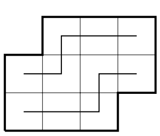
$M(2,6)$



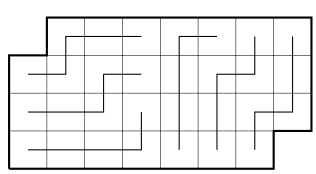
$M(2,11)$



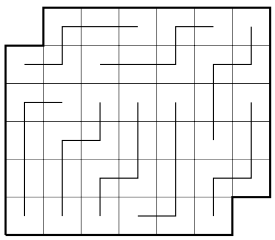
$M(3,4)$



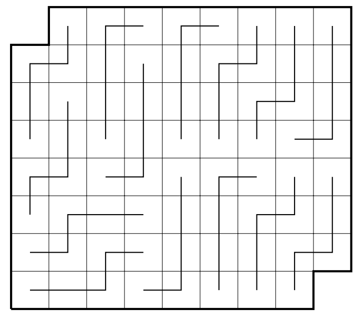
$M(4,8)$



$M(6,7)$

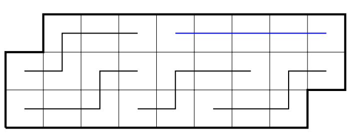


$M(8,9)$

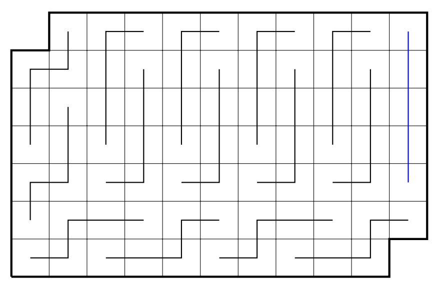


For each of our tiled base cases, $ab \equiv 2 \pmod{10}$. We show below that of our last four base cases with the right area, two of them are tiled by an odd number of height-0 tiles in T_5. Thus by Lemma 1 and our Symmetry Lemma, all four of these regions are not tiled:

$M(3,9)$



$M(7,11)$



For each of these untiled base cases, $ab \equiv 7 \pmod{10}$. Now that we have all of our base case regions identified as either tiled or untiled, we can use our lemmas as an inductive step to deduce all other possible regions. By Lemma 3, if $ab \equiv 7 \pmod{10}$, then $M(a,b)$ is not tiled by T_5'. By Lemma 2, $M(a,b)$ is tiled if and only if $ab \equiv 2 \pmod{10}$ for all $a, b > 1$.

□

4 Connections & Contributions

Igor Pak determines which $[a \times b]$ rectangles are tiled by T_5' below:

Theorem 2. ***Theorem 0.1.*** *If an* $[a \times b]$ *rectangle can be tiled by height-1 ribbon tile pentominoes, then* $10|a \cdot b$

Our results contribute a satisfying equivalent property to Pak's. That is, just as tiled $[a \times b]$ rectangles have area divisible by 10, so do tiled modified rectangles using the set T_5'. As stated in our **Tileability Theorem for** T_5': T_5' tiles $M(a,b)$ if and only if $ab \equiv 2 \pmod{10}$. Thus $M(a,b)$ is tiled if and only if $10|ab-2$. Our results and Pak's results also both depend on the use of tile invariants to rule out untiled regions.

Example. The $[3 \times 10]$ rectangle in Figure 18 is tiled by T_5' since its area is divisible by 10. All rectangles with area not divisible by 5 can be ruled out by our area invariant. For example, the $[4 \times 6]$ rectangle in Figure 19 cannot be tiled.

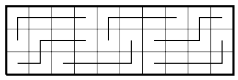


Figure 18, $[3 \times 10]$ Rectangle

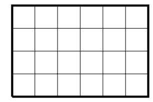


Figure 19, $[4 \times 6]$ Rectangle

Example. The $[7 \times 5]$ rectangle is tiled by 1 height-0 tile in T_5. By our Tileability Test, this region is not tiled by T_5'.

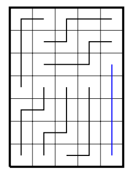


Figure 20, $[7 \times 5]$ Rectangle

Also important to note, our inspiration and approach to this tiling problem parallels Hitchman's. Hitchman proves which modified rectangles are tiled by the set T_4' (here he denotes this set of height-1 tetrominos as S):

Theorem 3. *Let $a, b > 1$. The set S tiles $M(a, b)$ if and only if*

1. $a \equiv 2 \pmod 4$ *and b is odd; or*
2. *a is odd and* $ab \equiv 2 \pmod 8$.

He uses area and height invariants to rule out untiled base-case $M(a, b)$'s, and also uses inductive and symmetry lemmas to prove all tiled and untiled $M(a, b)'s$. Hitchman's work also builds off of Pak's discovery of which $[a \times b]$ rectangles are tiled by the set T_4'.

Example. The region $M(6, 3)$ is tiled by T_4'. We can use our area invariant to rule out all other regions with area not divisible by 4. For example, $M(2, 4)$ is not tiled.

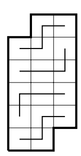


Figure 21, $M(6, 3)$

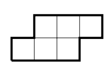


Figure 22, $M(2, 4)$

Example. Region $M(7, 2)$ is tiled by 3 height-0 tiles in T_4. By our Tileability Test, $M(7, 2)$ is not tiled by T_4'.

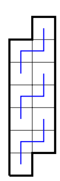


Figure 23, $M(7, 2)$

5 Future Directions

This study incorporates historically developed tiling invariants to prove the nonexistence of tilings. Its inclusion of the Conway/Lagarias and height invariants, as well as its contributions to Pak's research on the set T_5', highlights the significance and beauty behind the re-occurring patterns we see in the mathematics of tiling.

This work is not yet finished, however. Many future explorations could be done regarding tiled regions by T_5'. One could study which squares and modified squares are tiled by height-1 ribbon tile pentominoes. One could even shift their focus from the euclidean plane to tiling regions in the hyperbolic plane. It would be fascinating to derive tile invariants in hyperbolic geometry. Expanding this study beyond the bounds of the euclidean world would be an intriguing endeavor, one that surely has been or soon will be explored.

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