SANDCASTLES

A MATH MODELING ADVENTURE

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WHAT IS THE MATH MODELING COMPETITION?

THE CHALLENGE (1 OF 5)

- 1. Construct a mathematical model to identify the best 3-dimensional geometric shape to use as a sandcastle foundation that will last the longest period of time on a seashore that experiences waves and tides under the following conditions:
  - built at roughly the same distance from the water on the same beach, and
  - built using the same type of sand, roughly the same amount of sand, and the same water-to-sand proportion.

ASSUMPTIONS

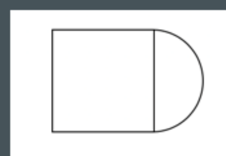
- The beach is uniform with a constant slope down to the water, and the coastline is a straight line.
- The waves at this beach are average and consistent.
- The rate of erosion of a sandcastle is dependent upon the distance to the average position of the waves.
- Water slowly erodes the sandcastle when it is in contact.
- Water can be modeled as a bunch of tiny spheres rolling up and down the beach.
- The only force on the water particles on the beach is gravity.
- Sand can be uniformly compacted.
- Rain can also be modeled as a bunch of tiny spheres.

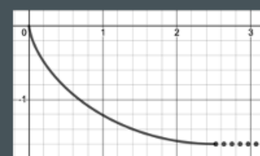
EARLY MODEL

- To describe erosion initially, we used $\frac{\partial y(x,t)}{\partial t} = -\frac{1}{W-y}$, where the y-direction is perpendicular to the ocean and W is the average position of the waves.
- The solution to this equation is:
  - $y(x,t) = W - \sqrt{W^2 + 2t - 2Wy_0(x) + y_0(x)^2}$
- Conclusions:
  - Want as much of the castle as far from the waves as possible (narrow is better)
  - For consistency of erosion make ocean-facing edge flat

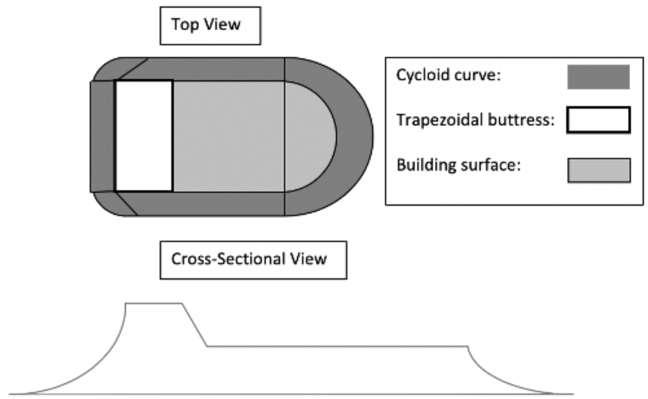
LATER MODEL

- Inspiration from bridge piers and seawalls
- Maximize surface to build the biggest sandcastle
- One side flat to reduce erosion
- Square-semi-circle combo
- Brachistochrone curve (cycloid) to reduce erosion





SOLUTION


Top View
Cycloid curve:
Trapezoidal buttress:
Building surface:
Cross-Sectional View

THE CHALLENGE (2 OF 5)

- 2. Using your model, determine an optimal sand-to-water mixture proportion for the castle foundation, assuming you use no other additives or materials (e.g. plastic or wooden supports, stones, etc.).

SOLUTION TYPE OF SAND AND WATER:SAND

- Utilize sand formed from glacial sheets moving over rock.
  - This type of sand has very jagged edges which increases the friction between sand particles and thereby making it easier for the sand to “catch” on one another
  - Consider using finer sand particles.
    - Easier to compact the sand together since there is a decrease in surface area and porosity.
- Optimal sand to water ratio is 1 parts water to 99 parts dry sand.
  - Ensures that there is enough water to provide a “bridge” between particles to bond them together, and not be too saturated or too dry.

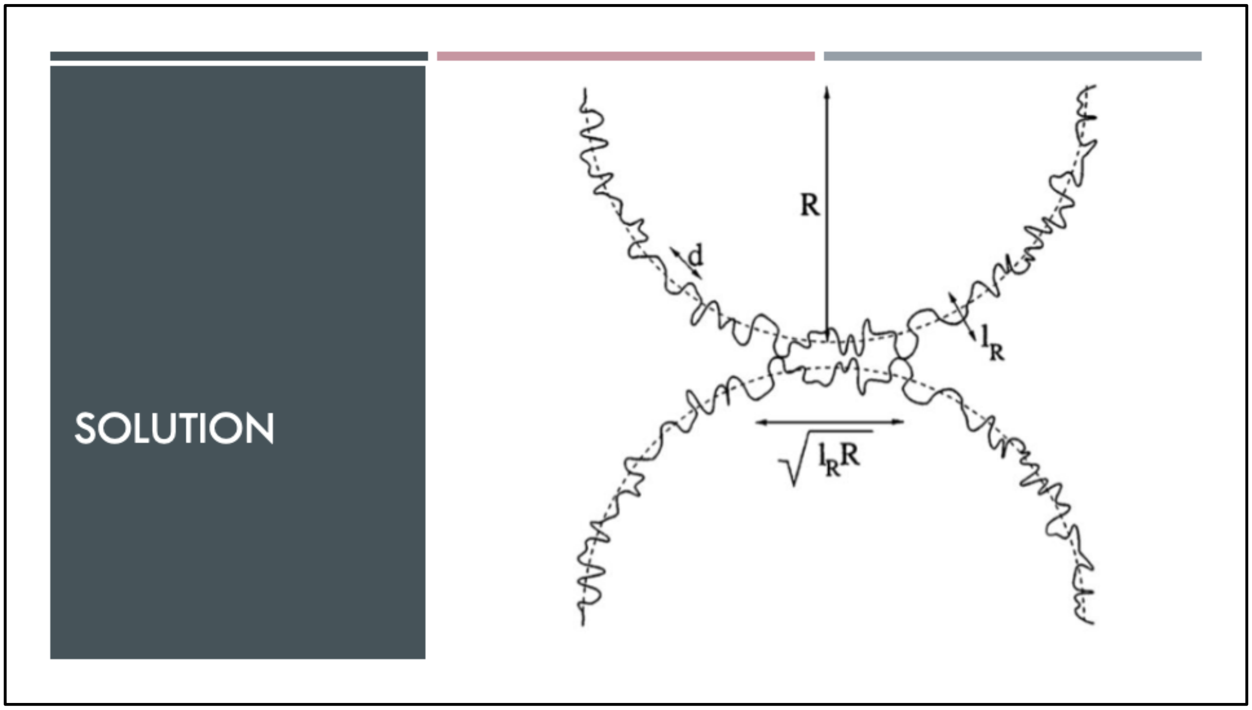
Australian Academy of Science. "Build it Up-Making the Perfect Sandcastle." https://www.science.org.au/curious/everything-else/sandcastles. (accessed 3/6/20)

Bonn D., Habibi M., Møller P., and Pakpour M. Nature. Scientific Reports. "How to Construct the Perfect Sandcastle." (accessed 3/6/20)

SOLUTION

- $G = 2(1 + \nu)\frac{F_{strain}/L^2}{\Delta x/L}$
  - Where *G* is the macroscopic shear modulus, *L* is the macroscopic cube dimension, ν is the Poisson ratio (≈ 0.5), $Fstrain/L2$, is the stress, and $\Delta x/L$ is the strain.
- $G = \alpha a^{-1/3} E^{2/3} \gamma^{1/3}$
  - Where α is the proportionality constant for how much the individual bonds are deformed, *a* is the radius of the grains, *E* is Young's modulus (stiffness of a material), and γ is the surface tension between liquid (sea water), and air.

Halsey T.C. and Levine A.J. Physical Review Letters. "How Sandcastles Fall." https://journals.aps.org/prl/pdf/10.1103/PhysRevLett.80.3141. (accessed 3/6/20)



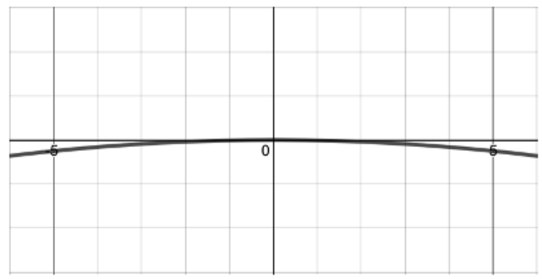
Demonstration of the contact of two not uniformed sand particles. Where *lR* is the average deviation in height, *R* is the radius of the particle, *d* is the distance between the height deviations, and $\sqrt{l_R R}$ describes the "macroscopic curvature" is not evident for where the particles are in contact. Reproduced from "How Sandcastles Fall", by Halsey T.C. and Levine A.J., Physics Review Letters (1998), Vol 80 Number 14, pg 3143.

THE CHALLENGE (3 OF 5)

- 3. Adjust your model as needed to determine how the best 3-dimensional sandcastle foundation you identified in requirement 1 is affected by rain, and whether it remains the best 3-dimensional geometric shape to be used as a castle foundation when it is raining.

SOLUTION

- Inspiration from soccer and football fields
- Pitch the flat surface 1-2% to allow rain to run off



THE CHALLENGE (4 OF 5)

- 4. What other strategies, if any, might you use to make your sandcastle last longer?

FUTURE WORK

- A few days later, after the storm passes, the boy comes back to the beach to make an even better sandcastle. He allows his imagination to go beyond what he had previously considered. He was very careful to choose the optimal location and environment to benefit the sandcastle's structural integrity.
- Dr. Daniel Bonn, University of Amsterdam: areas of high air humidity and low temperature prevent the evaporation of the water holding the granular sand particles together ("How To Build The Perfect Sandcastle, According to a Physicist") Hence, the boy found a beach that experiences colder, rainier weather, as warmer, drier conditions could lead to the entire structure (the sandcastle, and its foundation) drying up and losing its consistency.

FUTURE WORK

- Other alternative strategies:
- Applying corn starch and/or glue to the mixture. Corn starch is often used as a thickening agent and can make the sand more compact. Also, both compounds provide more adhesion between the granular particles in the structure itself.
- Constructing sea wall from rock/stone covered in wet sand. Large rocks stacked on top of each other make the structure less likely to move and would be able to break the incoming tide and reduce the amount of exposure of seawater on the foundation, even if the sand is removed over time by erosion.
- A hole is dug in the center of the foundation, with an umbrella installed to serve as a canopy for the sandcastle. A plastic-lined moat encircles the sandcastle to collect ocean and rainwater and prevent it from washing away the structure.
- A small pump installed in the trough of the moat to divert water to small mist filters on the umbrella. The filters are distributed in such a way to keep a constant 1:99 water-to-sand ratio and avoid any event of over-saturation.

THE CHALLENGE (5 OF 5)

- 5. Finally, write an informative, one- to two-page article describing your model and its results for publication in the vacation magazine: *Fun in the Sun,* whose readers are mainly non-technical.

"FUN IN THE SUN" ARTICLE:
WHAT DOES IT TAKE TO BUILD "THE ULTIMATE SANDCASTLE"?

- What if you could build a sandcastle that could stand the test of time? Michael Herman, age 13, has taken on the challenge. Herman chose the sandcastle as the topic of a project to be submitted into the Science, Technology, Engineering, and Mathematics (or STEM) Expo at his middle school, Yamhill-Carlton Intermediate School.
- The problem that concerned Herman the most was the impact of tidal waves. Thus, the castle was not directly aligned with the ocean, as it was elevated on a foundation to avoid exposure to seawater. (Water could soften the sand and make the castle fall apart.)
- The sea-facing side of the castle featured a sea wall that rose above the castle height, acting as a barrier against the incoming tides, and also a wind-breaking structure. The foundation itself was curved upwards to allow gravity to slow down the incoming tide and prevent it from reaching the castle.
- The curved corners of the foundation redirected water around the castle and minimized the amount of sand removed from the base. Instead of a sea wall for the back of the castle, Herman carved a curved edge in favor of a flat edge to reduce water erosion.
- The castle lasted for two days, according to Herman. When asked if he would make the sandcastle again, and what he would do differently, Herman said yes, but noted that he would use more materials, and not just wet sand.

QUESTIONS?