Complete Multipartite Graphs and the Relaxed Coloring Game

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The Coloring Game

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Two players, Alice and Bob, alternate coloring the verticies of a finite graph G with legal colors from a set X of r colors. In the d-relaxed coloring game a color $\alpha \in X$ is legal for an uncolored vertex v if v has at most d neighbors previously colored α. Alice wins the d-relaxed coloring game if every vertex in the graph is colored. Otherwise, Bob wins if there comes a point in the game where there is no legal color for a vertex. The least r for which Alice has a winning strategy in the d-relaxed coloring game is called the d-relaxed game chromatic number and is denoted ${}^d\chi_g(G)$.

Developments

History

A complete multipartite graph is a graph whos vertices can be placed into independent sets such that if P_i and P_j are distinct partite sets then each vertex in P_i is adjacent to each vertex in P_j. A complete multipartite graph is a complete equipartite graph if all P_i and P_j are equipollent.

The following theorems bound the 0- and 1-relaxed game chromatic numbers for complete equipartite graphs.

Theorem 1. *(Dunn, 2011)*
Let r and n be positive integers. If $G = K_{r*n}$, then

$$\chi_g(G) = \begin{cases} r & \text{if } n = 1 \\ 2r-2 & \text{if } n = 1 \text{ and } r \geq 3 \\ 2r-1 & \text{otherwise} \end{cases}$$

Theorem 2. *Let r and n be positive integers with $r \geq 2$. If $G = K_{r*n}$, then ${}^1\chi_g = \lceil \frac{rn}{2} \rceil$.*

The classification of the game chromatic number based on the size of the partite set begs the question:

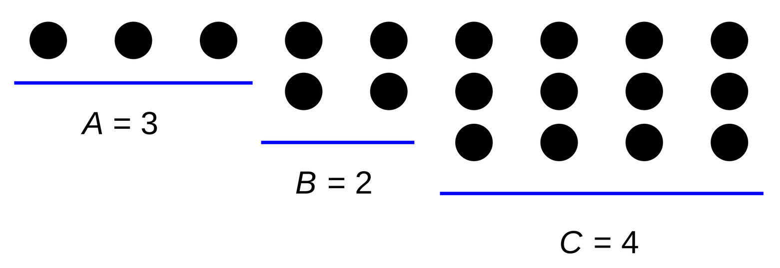
Question. *What is ${}^d\chi_g(G)$ if each partite set is not equipollent?*

For the 0-relaxed coloring game, the $\chi_g(G)$ varies depending on the number of sets of a particular size.

Definitions

Definition. *The classification of the partite set sizes is as follows:*

1. $A = |\{P_i : |P_i| = 1 \text{ and } i \in 1, 2, \ldots, n\}|$
2. $B = |\{P_i : |P_i| = 2 \text{ and } i \in 1, 2, \ldots, n\}|$
3. $C = |\{P_i : |P_i| = 3 \text{ and } i \in 1, 2, \ldots, n\}|$
4. $D = |\{P_i : |P_i| \geq 4 \text{ and } i \in 1, 2, \ldots, n\}|$



5. $|G| = \sigma$ is the total number of verticies in the graph.
6. A graph G is semi-Hamiltonian if some subgraph of G is a spanning path.

Main Results

The 0-relaxed Coloring Game

The following are a few of the man theorems concerning the 0-relaxed game.

Theorem 3. *If G is a complete multipartite graph with $B = D = 0$ and $A, C > 0$ then,*

$$\chi_g(G) = \begin{cases} A + 2C & \text{if } A > C + 1 \\ A + 2C - 1 & \text{if } A = C + 1 \text{ or } A = C \\ A + 2C - 2 & \text{if } A < C \end{cases}$$

Theorem 4. *If G is a complete multipartite graph with $A, B > 0$ and $C = D = 0$ then,*

$$\chi_g(G) = \begin{cases} A + B & \text{if } A \text{ is odd} \\ A + 2B - 1 & \text{if } A \text{ is even} \end{cases}$$

For all $A, B, C > 0$ and $D = 0$, the 0-relaxed game chromatic numbers are now known.

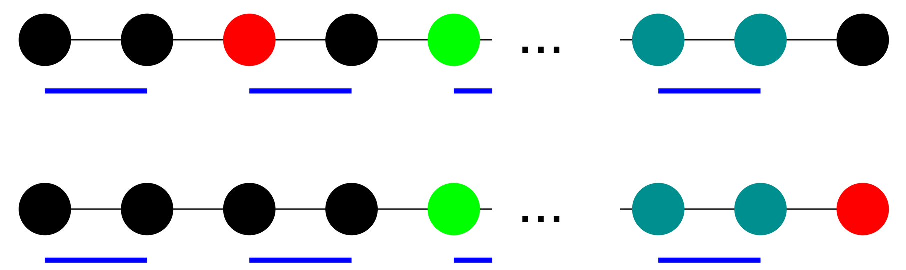
The 1-relaxed Coloring Game

Theorem 5 (Barrett, Portin, Sistko 2012). *If G is a complete multipartite, semi-Hamiltonian graph then ${}^1\chi_g(G) = \lceil \frac{\sigma}{2} \rceil$. Otherwise, if ψ is the number of vertices in $P_1, P_2, \ldots, P_{n-1}$, then ${}^1\chi_g(G) = \psi + 1$.*

Proof. We consider the case for odd σ. Let $H = v_1 \ldots v_n$ be a semi-Hamiltonian path in G. We claim that at the end of each of Bob's turns, there is at most one remote vertex in H. Note that the first remote vertex will be created if after Alice colors v_{n-2}, Bob colors v_{n-1} by his strategy. Then Alice can either color v_n or a different vertex.

Suppose she colors some vertex $v_i \neq v_n$. Then there are at most two isolated vertices in H, and Bob can color in one of them. Otherwise Alice colors v_n, and there are an even number of uncolored vertices in H. If v_n was the last vertex, we are done; otherwise, there are at least two uncolored vertices in H. Since v_n was the only remote vertex in H, each uncolored vertex is necessarily the member of a pair. Since v_n cannot be in the same partite set as both members of a pair, then at least one member of a pair is in a partite set different from the partite set of v_n. Bob can then color this vertex.

Bob maintains this until all but one vertex is colored. Since each color is used at most twice, at least $\lfloor \frac{\sigma}{2} \rfloor$ colors have been used. Hence, Bob wins if only that many colors are available, so that ${}^1\chi_g(G) \geq \lceil \frac{\sigma}{2} \rceil$. □



Classification

Semi-Hamiltonicity

Theorem 6. *If G is a complete multipartite graph then order $|P_i|$ to form a nondecreasing sequence $|P_1|, |P_2|, \ldots, |P_n|$. Then,*

1. *G is semi-Hamiltonian if $|P_n| \leq \lceil \frac{\sigma}{2} \rceil$.*
2. *G is not semi-Hamiltonian if $|P_n| > \lceil \frac{\sigma}{2} \rceil$.*

Proof. This result is easily shown by construction. Starting in P_n, add vertices to a list H, which will be a path in G, by alternating between vertices in P_n and vertices in the remaining partite sets. Then, if P_n has more than $\lceil \frac{\sigma}{2} \rceil$ vertices, when each vertex in $P_1, \ldots, P_{n-1}$ is be in H, only vertices in P_n will remain. Notice, this was the optimal way to account for vertices in P_n. The case where P_n has less than $\lceil \frac{\sigma}{2} \rceil$ is left without proof. □

Thus we have shown what the 1-relaxed game chromatic number of a complete multipartite graph is in terms of its semi-Hamiltonicity, and classified when a complete multipartite graph will be semi-Hamiltonian.

Open Questions

Question. *The cases for A, B, and $C > 0$ have been considered with $D = 0$. What would happen to $\chi_g(G)$ if $D > 0$?*

Question. *Although the ${}^0\chi_g(G)$ and ${}^1\chi_g(G)$ have been shown for complete multipartite graphs, the ${}^d\chi_g(G)$ for $d > 1$ is still unknown.*

Question. *For each non-negative integer d does there exist a graph G so that ${}^d\chi_g(G) \leq {}^{d+1}\chi_g(G)$?*

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