Asymptotic Behavior of Travelling Wave Solutions to Reaction-Diffusion Equations

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Abstract

We will discuss travelling wave solutions to reaction-diffusion equations of the form:

$$u_t = u_{xx} + u^p(1 - u^q) \quad (1)$$

which can be used as a mathematical model for various biological phenomena, as well as to model problems in combustion theory. We identify conditions on the wave speed so that travelling wave solutions exist for the case p ≥1 and q ≥1. Moreover, we estimate the rate of decay of the travelling wave solutions. When p > 1 and q ≥1, this estimate requires center manifold theory because the typical linear methods fail to work. Through the mathematical analysis of reaction diffusion equations, the results of this research create further studies and application in physical and industrial chemistry.

Introduction

The Fisher-Kolmogoroff equation

$$u_{tt} = u_{xx} + u(1-u) \quad (2)$$

was proposed in 1937 as a model for the spread of a favored gene in a population. This equation is an extension of the logistic growth equation where the population disperses through diffusion. We are motivated to seek travelling wave solutions to equation (1) as it is a generalization of the Fisher-Kolmogoroff equation. This information can be used as a model for the ignition phase in solid fuel combustion, chemical kinetics, spread of a favored gene, and many other physical phenomena.

For our purposes, a travelling solution is a solution that travels with a constant shape.

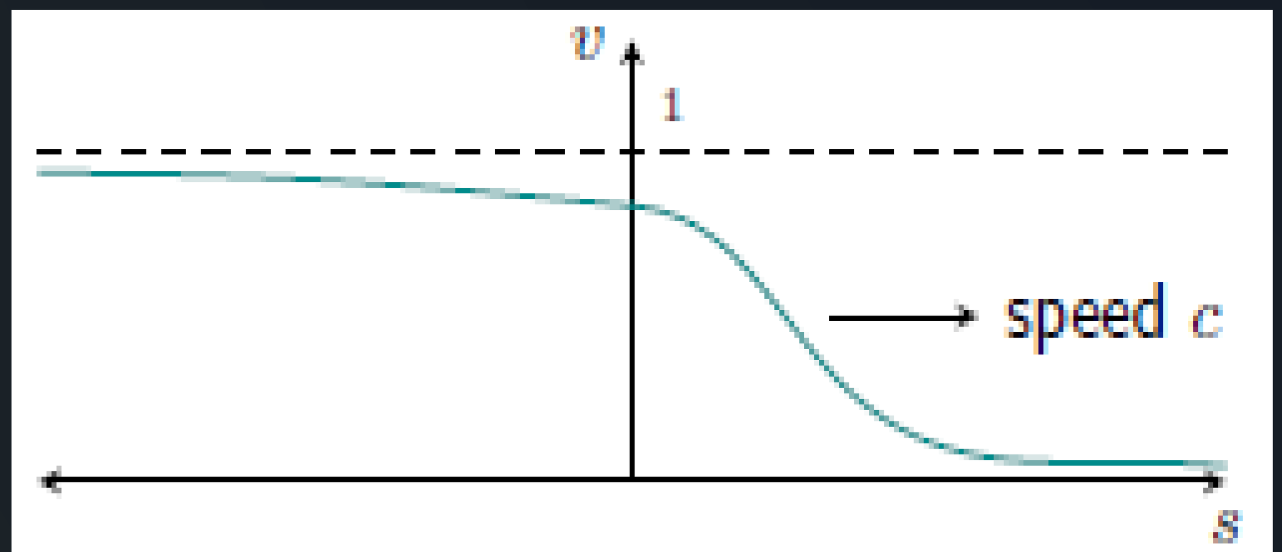


Figure 1: Generic travelling wave solution propagating with a constant shape.

Mathematical Analysis

We consider the reaction-diffusion equation after a suitable rescaling of t, x, and u:

$$\begin{cases} u_t = u_{xx} + u^p(1-u^q), & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (3)$$

Does there exist travelling wave solutions

$$u(x,t) = v(s), \qquad s = x - ct,$$

Where $c > 0$ is the wave speed and

$$v(s) \longrightarrow 1 \quad as \quad s \longrightarrow -\infty$$
$$v(s) \longrightarrow 0 \quad as \quad s \longrightarrow \infty$$

The desired travelling wave solution corresponds to a heteroclinic orbit in the phase portrait with α-limit set (1,0) and ω-limit set (0,0).

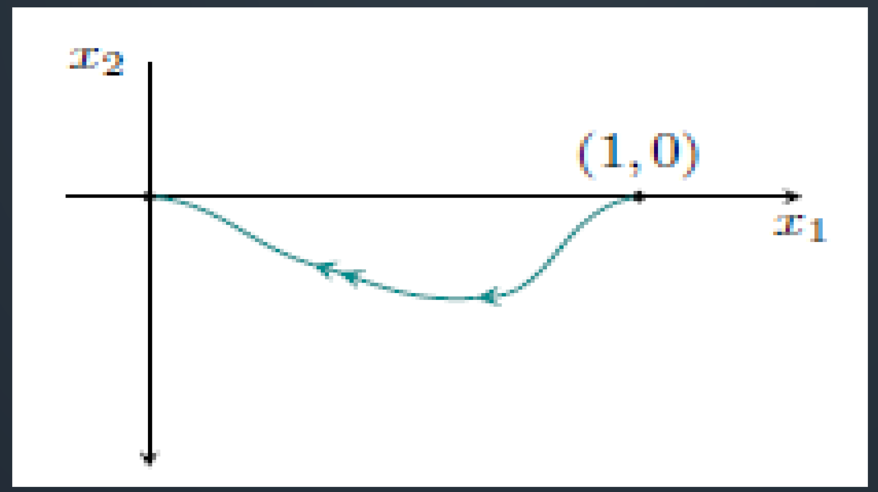


Figure 2: Heteroclinic orbit with α-limit set (1,0) and ω-limit set (0,0).

We considered the relevant fixed points at (0,0) and (1,0). Linearization fails to predict the behavior near (0,0) but accurately predicts the behavior near (1,0). By Hartman-Grobman Theorem, the behavior near the fixed point (1,0) for the nonlinear system looks like:

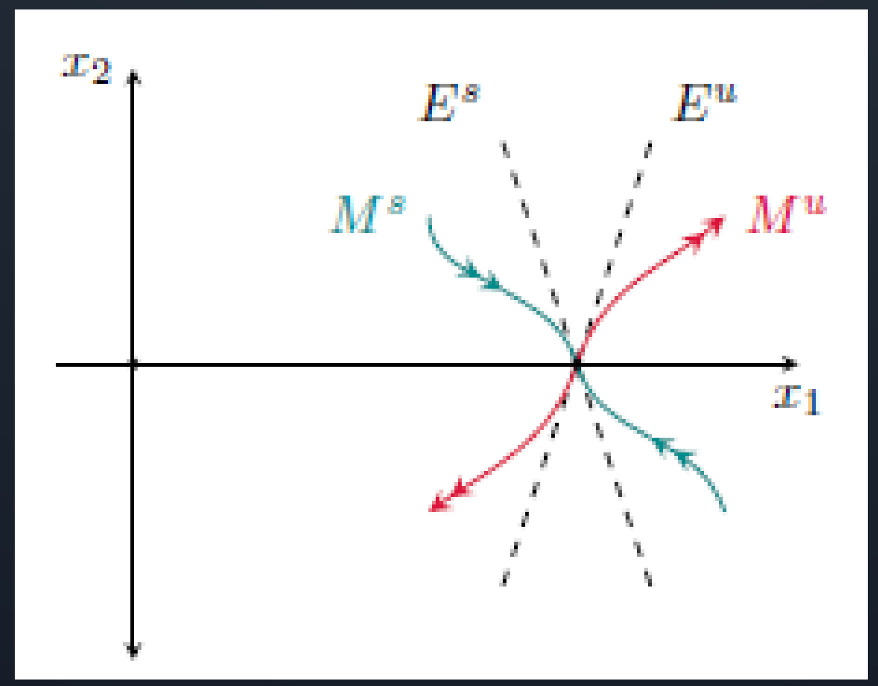
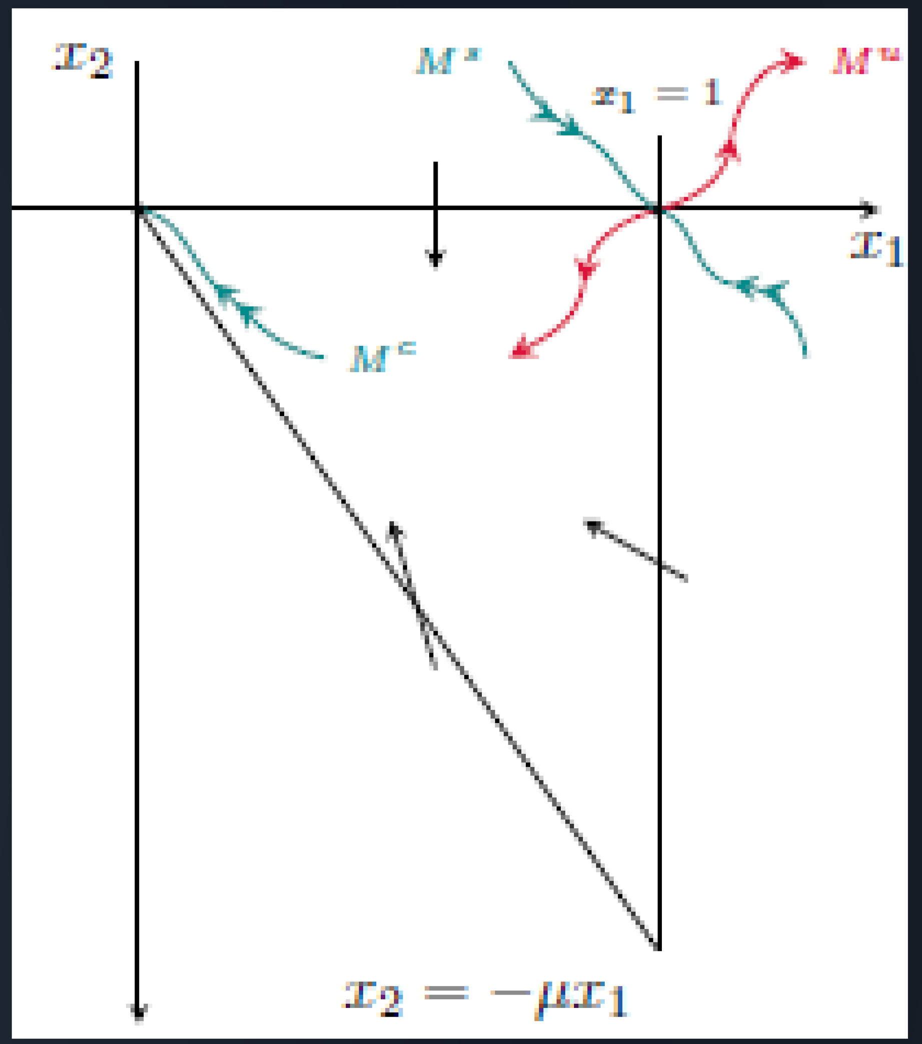


Figure 3: The behavior near the fix point (1,0) using the Hartman-Grobman Theorem.

Because linearization failed to predict the behavior of the fixed point (0,0), we investigated the local structure using center manifold theory. From center manifold theory, we observed that local structure near the origin in the fourth quadrant had the desired behavior corresponding to desired heteroclinic orbit.

Mathematical Analysis Cont.



We were then motivated to seek a trapping region with the structure seen in figure 4. By creating a trapping region with the slope of the diagonal line less than the slope of the flow, we were able to verify the existence of the desired heteroclinic orbit, thus verifying the existence of a travelling wave solution to the original reaction-diffusion equation.

Figure 4: Trapping region that contains desired heteroclinic orbit. Note: along the line $x_2 = -ux_1$, the slope of the flow is steeper than the slope of the line.

Results

From our constructed trapping region, it is observed that a heteroclininc orbit with the desired properties exists from (1,0) to (0,0). This heteroclinic orbit cannot touch the boundaries of our trapping region due to uniqueness and continuous dependence. This implies the original PDE has a travelling wave with speed

$$c^2 \geq \frac{4q(p-1)^{\frac{p-1}{q}}}{(p+q-1)^{\frac{p+q-1}{q}}} \quad (4)$$

We estimated the rate of decay of the heteroclinic orbit using center manifold stability theorem. Therefore the travelling wave solution v(s) will have the asymptotic behavior

$$v(s) = x_1(s) = y_1(s) - \frac{1}{c} y_2(s) \quad (5)$$

as $s \longrightarrow \infty$.

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