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HIGHER DIMENSIONAL LATTICE CHAINS AND DELANNOY NUMBERS

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Abstract. Fix nonnegative integers $n_1, \ldots, n_d$, and let L denote the lattice of points $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ that satisfy $0 \leq a_i \leq n_i$ for $1 \leq i \leq d$. Let L be partially ordered by the usual dominance ordering. In this paper we use elementary combinatorial arguments to derive new expressions for the number of chains and the number of Delannoy paths in L. Setting $n_i = n$ (for all i) in these expressions yields a new proof of a recent result of Duichi and Sulanke [9] relating the total number of chains to the central Delannoy numbers. We also give a novel derivation of the generating functions for these numbers in arbitrary dimension.

1. Introduction

Lattice chains and Delannoy paths have commanded a great deal of attention historically, and have enjoyed a surge of interest in recent decades. Popular expositions of the subject, like Comtet [8] and Stanley [15], have certainly given further impetus to their study, while also providing powerful tools for their analysis. At the same time, interest has also been generated with the appearance of connections to other topics, as chains have been studied in relation to simplicial complexes, Legendre polynomials, formal languages, ballot numbers, and probability theory, to name only a few [11, 6, 14, 4, 3]. The interested reader can find even more on these topics in the survey by Banderier and Schwer [3], with over 75 bibliographic references.

A particular charm of the topic is the interplay between counting arguments and generating function techniques. In Stanley [15], a problem involving lattice chains and Delannoy paths in two dimensions was used to illustrate a technique for extracting the diagonal of a generating function. Specifically, in the special case when the 2-dimensional lattice is square, the number of chains exceeds the number of Delannoy paths by a factor of an appropriate power of 2. The question was then posed to find a combinatorial proof of the same result.

This challenge was met by Sulanke in [16], who established a bijective correspondence by composing a sequence of intermediate bijections between six different step sets in the 2-dimensional lattice for the central (diagonal) case. More recently, his article with Duchi [9] generalizes the result to the central case in arbitrary dimension, again by means of a composition of explicit bijections. In the present paper, we offer elementary counting techniques that yield a number of new expressions, both for chains and Delannoy paths in the general (not necessarily central) lattice, in any dimension. The expressions for the chains and for the Delannoy numbers are strikingly similar to each other, and upon an appropriate subtitution, the central

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lattice is obtained as a special case, yielding an alternate proof of Sulanke's theorem in any dimension.

The chain numbers and the Delannoy numbers satisfy similar cross-dimensional recurrence relations, and we exploit this recursive structure to prove a result which generalizes both recurrences and offers a new means to derive their generating functions easily and uniformly in any dimension.

The paper is organized as follows. Section 2 fixes notation and describes our results on lattice chains, including k-chains, reducible chains, and finally the total number of chains. In Section 3, we consider Delannoy paths, first with k steps, and then the general case, obtaining the desired expression related to the number of chains. Finally, in Section 4, we introduce the class of $\mathfrak{a}$-recurrent sequences of functions – a class that includes both the chain numbers and the Delannoy numbers as special cases – and we offer an explicit expression for their generating functions in any dimension.

2. Results on Lattice chains

Throughout this paper, $\mathbb{N}$ denotes the nonnegative integers and $\mathbb{P}$ the positive integers. Fix $d \in \mathbb{P}$ and $\mathbf{n} \in \mathbb{N}^d$, where $\mathbf{n} = (n_1, \ldots, n_d)^T$. Let $L(\mathbf{n})$ denote the lattice of integer points $(a_1, \ldots, a_d)^T \in \mathbb{N}^d$ satisfying $a_i \leq n_i$ for $1 \leq i \leq d$.

Recall $L(\mathbf{n})$ is partially ordered by the dominance relation, defined as follows. Given $\mathbf{a}, \mathbf{b} \in L(\mathbf{n})$ with $\mathbf{a} = (a_1, \ldots, a_d)^T$ and $\mathbf{b} = (b_1, \ldots, b_d)^T$, we say $\mathbf{a} \preceq \mathbf{b}$ whenever $a_i \leq b_i$ for each i $(1 \leq i \leq d)$. We write $\mathbf{a} \prec \mathbf{b}$ whenever $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$.

Define the *weight* of an element $\mathbf{a} = (a_1, \ldots, a_d)^T \in L(\mathbf{n})$ by $\mathrm{wt}(\mathbf{a}) = a_1 + \cdots + a_d$. We define the *truncation* of $\mathbf{a}$ to be the $(d-1)$-tuple $\mathbf{a}' = (a_1, \ldots, a_{d-1})^T$.

2.1. Counting k-chains and some variations.

By a *chain* in $L(\mathbf{n})$ we mean a subset of $L(\mathbf{n})$ that is totally ordered by $\preceq$. A *k-chain* is a chain with k elements. Let $C(\mathbf{n})$ denote the set of chains in $L(\mathbf{n})$, and for each integer k, let $C_k(\mathbf{n})$ denote the set of k-chains in $L(\mathbf{n})$. In this section we study expressions for $|C_k(\mathbf{n})|$ and $|C(\mathbf{n})|$.

Expressions for $|C_k(\mathbf{n})|$ are not difficult to derive, and have been computed in several places [10, 13] for the special case $n_i = 1$ for all i, and, in [7], for the general case. Each of these derivations proceeds either by solving an appropriate recurrence or through the use of generating functions. In [5], a direct counting argument was given for $|C_k(\mathbf{n})|$ using the principle of inclusion/exclusion.

Lemma 1. [5],[7] *Fix $\mathbf{n} \in \mathbb{N}^d$, where $\mathbf{n} = (n_1, \ldots, n_d)^T$ and for each $k \in \mathbb{N}$, let $C_k(\mathbf{n})$ denote the set of k-chains in the corresponding lattice $L(\mathbf{n})$, and $\widetilde{C}_k(\mathbf{n})$ the set of chains in $C_k(\mathbf{n})$ that contain the maximum element $\mathbf{n}$. Then the following hold:*

(i) *The maximum length of a chain in $L(\mathbf{n})$ is given by*

$$k_{\max} = \mathrm{wt}(\mathbf{n}) + 1.$$

(ii) *For any integer k $(1 \leq k \leq k_{\max})$, the number of k-chains in $L(\mathbf{n})$ is given by*

$$|C_k(\mathbf{n})| = \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \prod_{i=1}^{d} \binom{n_i + k - r}{n_i}.$$

(iii) *For any integer k ($1 \le k \le k_{\max}$), the number of k-chains in $L(\mathbf{n})$ that contain $\mathbf{n}$ is given by*

$$|\widetilde{C}_k(\mathbf{n})| = \sum_{i=1}^{k} (-1)^{i+1} |C_{k-i}(\mathbf{n})|.$$

Proof. (i). If $\mathbf{a}$ and $\mathbf{b}$ are elements of $L(\mathbf{n})$ such that $\mathbf{a} \preceq \mathbf{b}$, then $0 \le \mathrm{wt}(\mathbf{a}) < \mathrm{wt}(\mathbf{b}) \le \mathrm{wt}(\mathbf{n})$. Since the weight of any element must be an integer, a chain can have at most $k_{\max} = \mathrm{wt}(\mathbf{n}) + 1$ elements. Conversely, a chain with length $k_{\max}$ can easily be defined inductively as follows. We simply set $\mathbf{a}_1 = \mathbf{0}$, and, given any $\mathbf{a}_i$ with $1 \le i \le k_{\max} - 1$, we define $\mathbf{a}_{i+1}$ by adding 1 to any coordinate a_{ij} of $\mathbf{a}_i$ for which $a_{ij} < n_j$.
(ii). An elementary proof using inclusion/exclusion is given in [5].
(iii). Note that $C_0(\mathbf{n}) = 1$ and $\widetilde{C}_0(\mathbf{n}) = 0$. For $k \ge 1$, each k-chain containing $\mathbf{n}$ corresponds to a unique $(k-1)$-chain that does not contain $\mathbf{n}$ (and conversely). So $|\widetilde{C}_k(\mathbf{n})| = |C_{k-1}(\mathbf{n}) \setminus \widetilde{C}_{k-1}(\mathbf{n})| = |C_{k-1}(\mathbf{n})| - |\widetilde{C}_{k-1}(\mathbf{n})|$. The result now follows by a simple induction. □

2.2. **Counting reducible chains.** We say a chain ξ is *reducible* if the truncations of its elements are pairwise distinct. Recall that chains are not defined as sequences, but as sub*sets* of the lattice, so a chain cannot contain repeated elements. With this in mind, we could equivalently define a k-chain ξ in $L(\mathbf{n})$ to be reducible iff the set ξ', formed by truncating the elements of ξ, remains a k-chain in $L(\mathbf{n}')$. For example, let $\mathbf{n} = (2,4,3,4)^T$ and suppose ξ_1 and ξ_2 denote the 3-chains

$$\xi_1 : \begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \end{pmatrix} \prec \begin{pmatrix} 0 \\ 3 \\ 2 \\ 2 \end{pmatrix} \prec \begin{pmatrix} 1 \\ 3 \\ 3 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \xi_2 : \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \prec \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} \prec \begin{pmatrix} 2 \\ 3 \\ 2 \\ 2 \end{pmatrix}.$$

Then ξ_1 is *not* reducible, since the first two elements have identical truncations. Equivalently, we could say that ξ_1 is not reducible since ξ_1' is only a 2-chain in $L(\mathbf{n}')$, as shown below. On the other hand, ξ_2 *is* reducible, since ξ_2' is still a 3-chain.

$$\xi_1' : \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \prec \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \xi_2' : \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \prec \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \prec \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

The next result is the analog of Lemma 1 for reducible chains.

Lemma 2. *With the notation of Lemma 1, let $C^{\mathrm{red}}(\mathbf{n})$ denote the set of reducible chains in $L(\mathbf{n})$, and let $\widetilde{C}^{\mathrm{red}}(\mathbf{n})$ denote the set of reducible chains that contain $\mathbf{n}$. Then the following hold.*

(i) *The maximum length of a chain in $C^{\mathrm{red}}(\mathbf{n})$ is*

$$k'_{\max} := \mathrm{wt}(\mathbf{n}') + 1.$$

(ii) *The number of reducible chains in $L(\mathbf{n})$ is*

$$|C^{\mathrm{red}}(\mathbf{n})| = \sum_{k=0}^{k'_{\max}} \binom{n_d + k}{n_d} |C_k(\mathbf{n}')|.$$

(iii) *The number of reducible chains in $L(\mathbf{n})$ that contain $\mathbf{n}$ is*

$$|\widetilde{C}^{\mathrm{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \binom{n_d + k - 1}{n_d} |\widetilde{C}_k(\mathbf{n}')|.$$

Proof. (i). By truncating the d-coordinates of each element, every reducible k-chain ξ in $L(\mathbf{n})$ corresponds to a unique k-chain ξ' in $L(\mathbf{n}')$. Therefore, $k \le k'_{\max}$ by Lemma 1(i).

(ii). Fix an integer k ($1 \le k \le k'_{\max}$) and let ξ be any reducible k-chain. Truncating the d-coordinates of the elements of ξ gives a unique k-chain ξ' in $L(\mathbf{n}')$, and the d-coordinates themselves form a non-decreasing sequence σ of integers between 0 and n_d (inclusive). Conversely, such a sequence and a k-chain in $L(\mathbf{n}')$ correspond to a unique reducible chain in $L(\mathbf{n})$. The number of such sequences is $\binom{n_d+k}{n_d}$. Multiplying by $|C_k(\mathbf{n}')|$ and summing over k, we obtain the result.

(iii). As in (ii) above, each ξ in $\widetilde{C}_k^{\mathrm{red}}(\mathbf{n})$ corresponds to a unique ξ' in $\widetilde{C}_k(\mathbf{n}')$ and a non-decreasing sequence σ of integers between 0 and n_d (inclusive), where σ contains n_d at least once. The number of such sequences is $\binom{n_d+k-1}{n_d}$. Multiplying by $|\widetilde{C}_k(\mathbf{n}')|$ and summing over k, we obtain the result. □

Corollary 1. *With the notation of Lemma 2, the number of reducible chains in $L(\mathbf{n})$ that contain $\mathbf{n}$ is given by*

$$|\widetilde{C}^{\mathrm{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \sum_{i=1}^{k} (-1)^{i+1} \binom{n_d + k - 1}{n_d} |C_{k-i}(\mathbf{n}')|.$$

Proof. Immediate by Lemma 1(iii) and Lemma 2(iii). □

By Lemma 1(ii), we can evaluate the term $|C_{k-i}(\mathbf{n}')|$ in the expression in Corollary 1 above to obtain a triple sum. As the next corollary shows, however, this reduces to a double sum.

Corollary 2. *With the notation of Lemma 2, the number of reducible chains in $L(\mathbf{n})$ that contain $\mathbf{n}$ is given by*

$$|\widetilde{C}^{\mathrm{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \sum_{i=1}^{k} (-1)^{i+k} \binom{k-1}{i-1} \binom{n_d + k - 1}{n_d} \prod_{j=1}^{d-1} \binom{n_j + i - 1}{n_j}.$$

Proof. Consider the expression for $|\widetilde{C}^{\mathrm{red}}(\mathbf{n})|$ given in Corollary 1 above. Recall that $|C_0(\mathbf{n})| = 1$, and for $i < k$ we can evaluate $|C_{k-i}(\mathbf{n})|$ using Lemma 1(ii) to obtain

$$(1) \quad |\widetilde{C}^{\mathrm{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \binom{n_d + k - 1}{n_d} \Bigg[(-1)^{k+1} + \sum_{i=1}^{k-1} \sum_{r=0}^{k-i-1} (-1)^{r+i+1} \binom{k-i-1}{r} \prod_{j=1}^{d-1} \binom{n_j + k - i - r}{n_j} \Bigg].$$

With the change of variables $r = k - i - t$, this simplifies to

(2) $$|\widetilde{C}^{\text{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \binom{n_d + k - 1}{n_d} (-1)^{k+1} \left[1 + \sum_{i=1}^{k-1} \sum_{t=1}^{k-i} (-1)^t \binom{k-i-1}{t-1} \prod_{j=1}^{d-1} \binom{n_j + t}{n_j} \right].$$

Interchanging the order of summation over i and t, this is equivalent to

(3) $$|\widetilde{C}^{\text{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \binom{n_d + k - 1}{n_d} (-1)^{k+1} \left[1 + \sum_{t=1}^{k-1} (-1)^t \prod_{j=1}^{d-1} \binom{n_j + t}{n_j} \sum_{i=1}^{k-t} \binom{k-i-1}{t-1} \right].$$

A common binomial identity [1, Thm. 1.8] states that $\sum_{i=1}^{k-t} \binom{k-i-1}{t-1} = \binom{k-1}{t}$. Applying this identity and then substituting $t = i - 1$, the bracketed expression simplifies to give the desired result. □

2.3. **The total number of chains.** Keeping with the notation of Lemma 2, we let $\widetilde{C}(\mathbf{n})$ denote the set of chains in $L(\mathbf{n})$ that contain $\mathbf{n}$. It is convenient to count $|\widetilde{C}(\mathbf{n})|$ rather than $|C(\mathbf{n})|$ directly. The difference is minimal, however, since removing $\mathbf{n}$ from each chain in $\widetilde{C}(\mathbf{n})$ gives a bijection between $\widetilde{C}(\mathbf{n})$ and $C(\mathbf{n}) \setminus \widetilde{C}(\mathbf{n})$, so that

(4) $$|C(\mathbf{n})| = 2 \cdot |\widetilde{C}(\mathbf{n})|.$$

Let $\mathcal{P}$ denote the power set of $\{0, 1, \ldots, n_d - 1\}$, and recall that $\widetilde{C}^{\text{red}}(\mathbf{n})$ denotes the set of reducible chains in $L(\mathbf{n})$ that contain $\mathbf{n}$. In this section we establish a bijection ϕ between $\widetilde{C}(\mathbf{n})$ and $\mathcal{P} \times \widetilde{C}^{\text{red}}(\mathbf{n})$.

Roughly speaking, ϕ can be described as follows. Given a chain ξ that contains $\mathbf{n}$, it fails to be reducible if the truncations of its elements are not distinct. The function ϕ removes from ξ any elements whose truncations are repeated by a later element in ξ. Doing so produces a reducible chain ξ^{red}. The d-coordinates of the elements removed are recorded in a set A_ξ. The output of ϕ is the pair $(A_\xi, \xi^{\text{red}})$.

More formally, we have the following.

Definition 1. *Suppose a chain ξ in $\widetilde{C}(\mathbf{n})$ has k elements $\mathbf{a}_1 \prec \cdots \prec \mathbf{a}_k$, where $\mathbf{a}_i = (a_{i1}, \ldots, a_{id})^T$ for each i $(1 \le i \le k)$. We define*

$$A_\xi = \{a_{id} \,|\, \mathbf{a}'_i = \mathbf{a}'_{i+1}\}, \qquad \textit{and} \qquad \xi^{\text{red}} = \xi \setminus \{\mathbf{a}_i \,|\, \mathbf{a}'_i = \mathbf{a}'_{i+1}\},$$

and we let $\phi(\xi)$ denote the pair $(A_\xi, \xi^{\text{red}})$. □

To illustrate this definition, let $\mathbf{n} = (3,3,3)^T$ and suppose ξ denotes the following 8-chain in $\widetilde{C}(\mathbf{n})$:

(5) $$\begin{matrix} & \mathbf{a}_1 & \prec & \mathbf{a}_2 & \prec & \mathbf{a}_3 & \prec & \mathbf{a}_4 & \prec & \mathbf{a}_5 & \prec & \mathbf{a}_6 & \prec & \mathbf{a}_7 & \prec & \mathbf{a}_8 \\ \xi : & \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & \prec & \begin{pmatrix} 2 \\ 1 \\ \mathbf{0} \end{pmatrix} & \prec & \begin{pmatrix} 2 \\ 1 \\ \mathbf{1} \end{pmatrix} & \prec & \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} & \prec & \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} & \prec & \begin{pmatrix} 3 \\ 2 \\ \mathbf{2} \end{pmatrix} & \prec & \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} & \prec & \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \end{matrix}.$$

Notice that $\mathbf{a}'_2 = \mathbf{a}'_3 = \mathbf{a}'_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{a}'_6 = \mathbf{a}'_7 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. The reducible chain ξ^{red} is formed by removing $\mathbf{a}_2$ and $\mathbf{a}_3$ (keeping $\mathbf{a}_4$), and removing $\mathbf{a}_6$ (keeping $\mathbf{a}_7$).

For each of the elements removed, their last coordinates (3rd coordinates in this case) are recorded in the set A_ξ. Then ξ^{red} is a reducible 5-chain in $\widetilde{C}^{\text{red}}(\mathbf{n})$, the set A_ξ is a subset of $\{0,1,2\}$, and $\phi(\xi)$ denotes the pair $(A_\xi, \xi^{\text{red}})$ below:
(6)

$$A_\xi = \{0,1,2\} \qquad \text{and} \qquad \xi^{\text{red}}: \begin{pmatrix}1\\1\\0\end{pmatrix} \prec \begin{pmatrix}2\\1\\2\end{pmatrix} \prec \begin{pmatrix}2\\2\\2\end{pmatrix} \prec \begin{pmatrix}3\\2\\3\end{pmatrix} \prec \begin{pmatrix}3\\3\\3\end{pmatrix}.$$

Observe that $\phi(\xi) \in \mathcal{P} \times \widetilde{C}^{\text{red}}(\mathbf{n})$.

Next we describe how the original chain ξ can be recovered from the pair $(A_\xi, \xi^{\text{red}})$. Given the information in line (6) above, we simply must reinsert into ξ^{red} the missing elements, one belonging to each member of A_ξ. Each x in A_ξ is the d-coordinate x_d of a point $\mathbf{x}$ that is to be inserted immediately to the left of the first $\mathbf{y}$ in ξ^{red} for which $x_d < y_d$. In our case, 0 and 1 belong left of $\begin{pmatrix}2\\1\\2\end{pmatrix}$, while 2 belongs left of $\begin{pmatrix}3\\2\\3\end{pmatrix}$:

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \prec \begin{pmatrix}\\\\\mathbf{0}\end{pmatrix} \prec \begin{pmatrix}\\\\\mathbf{1}\end{pmatrix} \prec \begin{pmatrix}2\\1\\2\end{pmatrix} \prec \begin{pmatrix}2\\2\\2\end{pmatrix} \prec \begin{pmatrix}\\\\\mathbf{2}\end{pmatrix} \prec \begin{pmatrix}3\\2\\3\end{pmatrix} \prec \begin{pmatrix}3\\3\\3\end{pmatrix}.$$

Observe that for each x in A_ξ, such a $\mathbf{y}$ is guaranteed to exist in ξ^{red} by the fact that every element of A_ξ is $< n_d$, while $\mathbf{n}$ belongs to ξ^{red}. Indeed, this motivates our choice to work with $\widetilde{C}(\mathbf{n})$ rather than $C(\mathbf{n})$. To complete the recovery of ξ, note that the remainder of each new point $\mathbf{x}$ is determined by the condition that $\mathbf{x}' = \mathbf{y}'$. In our case, 0 and 1 are topped by $\begin{pmatrix}2\\1\end{pmatrix}$, while 2 is topped by $\begin{pmatrix}3\\2\end{pmatrix}$. Doing so produces the original chain ξ, given in (5).

The next three lemmas establish the relevant properties of ϕ.

Lemma 3. *With the above notation, ϕ is a function from $\widetilde{C}(\mathbf{n})$ to $\mathcal{P} \times \widetilde{C}^{\text{red}}(\mathbf{n})$.*

Proof. For $\xi \in \widetilde{C}(\mathbf{n})$, recall $\phi(\xi) = (A_\xi, \xi^{\text{red}})$. If $x \in A_\xi$, then $x = a_{id}$ for some i, where $\mathbf{a}'_i = \mathbf{a}'_{i+1}$. But $\mathbf{a}_i \prec \mathbf{a}_{i+1}$, so $a_{id} < a_{i+1,d}$. Thus every element of A_ξ is strictly less than n_d, and $A_\xi \in \mathcal{P}$. To show $\xi^{\text{red}} \in \widetilde{C}^{\text{red}}(\mathbf{n})$, note that $\xi^{\text{red}} \subseteq \xi$, so ξ^{red} is totally ordered by $\prec$. And $\mathbf{n} \in \xi$ since $\xi \in \widetilde{C}(\mathbf{n})$, while $\mathbf{n} \notin \{\mathbf{a}_i \,|\, \mathbf{a}'_i = \mathbf{a}'_{i+1}\}$ so $\mathbf{n} \in \xi^{\text{red}}$. It remains to show ξ^{red} is reducible. Suppose there were $\mathbf{x} \prec \mathbf{y}$ in ξ^{red} such that $\mathbf{x}' = \mathbf{y}'$. Then $\mathbf{x} = \mathbf{a}_i$ and $\mathbf{y} = \mathbf{a}_j$ for some $i < j$. But $\mathbf{a}'_i \preceq \mathbf{a}'_{i+1} \preceq \mathbf{a}'_j$ so $\mathbf{a}'_i = \mathbf{a}'_{i+1}$ and thus $\mathbf{a}_i \notin \xi^{\text{red}}$, a contradiction. It follows that $\xi^{\text{red}} \in \widetilde{C}^{\text{red}}(\mathbf{n})$. □

Lemma 4. *With the above notation, ϕ is injective.*

Proof. Let ξ_1, ξ_2 be in $\widetilde{C}(\mathbf{n})$ and suppose $\phi(\xi_1) = \phi(\xi_2)$. Then $\xi_1^{\text{red}} = \xi_2^{\text{red}}$, and to prove $\xi_1 = \xi_2$, it remains to show that $\xi_1 \setminus \xi_1^{\text{red}} = \xi_2 \setminus \xi_2^{\text{red}}$. We accomplish this by proving that for any chain ξ in $\widetilde{C}(\mathbf{n})$, each element $\mathbf{x} \in \xi \setminus \xi^{\text{red}}$ corresponds to a unique element $x_d \in A_\xi$, and that, in fact, $\mathbf{x}$ can be explicitly constructed from the element $x_d \in A_\xi$ and the chain ξ^{red}. Performing this construction for each element of A_ξ then yields the entire set $\xi \setminus \xi^{\text{red}}$. To describe the construction, let $\mathbf{x}$ be any element of $\xi \setminus \xi^{\text{red}}$, and let k denote the length of ξ. Then $\mathbf{x} = \mathbf{a}_i$ for some i where $\mathbf{a}'_i = \mathbf{a}'_{i+1}$ and $x_d = a_{id} \in A_\xi$. Let $t = \max\{j \mid \mathbf{a}'_i = \mathbf{a}'_j\}$. Then $t \geq i+1$ and $\mathbf{a}'_i = \mathbf{a}'_{i+1} = \cdots = \mathbf{a}'_t$. Also, either $t < k$ and $\mathbf{a}'_t \neq \mathbf{a}'_{t+1}$ or else $t = k$ and $\mathbf{a}_t = \mathbf{n}$. In either case, $\mathbf{a}_t \in \xi^{\text{red}}$, and $\mathbf{a}_i, \ldots, \mathbf{a}_{t-1} \notin \xi^{\text{red}}$. So $\mathbf{a}_t = \min\{\mathbf{y} \mid \mathbf{y} \in \xi^{\text{red}} \text{ and } \mathbf{x} \prec \mathbf{y}\}$. Observe that since $\mathbf{x} \prec \mathbf{a}_t$ and $\mathbf{x}' = \mathbf{a}'_t$, it must be the case that $x_d < a_{td}$. It follows that $\mathbf{a}_t = \min\{\mathbf{y} \mid \mathbf{y} \in \xi^{\text{red}} \text{ and } x_d < y_d\}$. Since $\mathbf{x}' = \mathbf{a}'_t$ and has d-coordinate x_d, we have now shown that $\mathbf{x}$ is completely determined by the element x_d in A_ξ and the chain ξ^{red}. It follows that ξ is determined by the pair $(A_\xi, \xi^{\text{red}})$, so ϕ is injective. □

Lemma 5. *With the above notation, ϕ is surjective.*

Proof. To see that ϕ is surjective, we associate a chain in $\widetilde{C}(\mathbf{n})$ with each pair (A, ζ) in $\mathcal{P} \times \widetilde{C}^{\text{red}}(\mathbf{n})$. Let (A, ζ) be such a pair and suppose ζ has t elements $\mathbf{b}_1 \prec \cdots \prec \mathbf{b}_t$ where $\mathbf{b}_i = (b_{i1}, \ldots, b_{id})^T$ for each i $(1 \leq i \leq t)$. For each x in A, define $m := \min\{j \mid x < b_{jd}\}$ and set $\mathbf{b}_x = (b_{m1}, \ldots, b_{m(d-1)}, x)^T$ in $L(\mathbf{n})$. In other words, we define $\mathbf{b}_x$ by putting $\mathbf{b}'_x := \mathbf{b}'_m$ and setting the d-coordinate of $\mathbf{b}_x$ equal to x. Then the chain in $\widetilde{C}(\mathbf{n})$ that we associate with the pair (A, ζ) is simply $\xi_{(A,\zeta)} := \zeta \cup \{\mathbf{b}_x \mid x \in A\}$. It is easy to check that $\phi(\xi_{(A,\zeta)}) = (A, \zeta)$ as desired. □

Corollary 3. *With the above notation, the map ϕ is a bijection between $\widetilde{C}(\mathbf{n})$ and $\mathcal{P} \times \widetilde{C}^{\text{red}}(\mathbf{n})$.*

Proof. Immediate from Lemmas 3-5. □

Theorem 1. *Fix $\mathbf{n} \in \mathbb{N}^d$ and let $C(\mathbf{n})$ denote the set of chains in $L(\mathbf{n})$. Then*

$$|C(\mathbf{n})| = 2^{n_d+1} \sum_{k=1}^{k'_{\max}} \sum_{i=1}^{k} (-1)^{i+k} \binom{k-1}{i-1} \binom{n_d+k-1}{n_d} \prod_{j=1}^{d-1} \binom{n_j+i-1}{n_j}.$$

Proof. By equation (4) and Corollary 3, we have $|C(\mathbf{n})| = 2 \cdot |\mathcal{P} \times \widetilde{C}^{\text{red}}(\mathbf{n})| = 2 \cdot |\mathcal{P}| \cdot |\widetilde{C}^{\text{red}}(\mathbf{n})|$. Since $|\mathcal{P}| = 2^{n_d}$, the result follows by Corollary 2. □

3. Results on Delannoy numbers and the Theorem of Sulanke

The set $D = D(\mathbf{n})$ of (generalized) Delannoy paths contains precisely those chains in $L(\mathbf{n})$ that contain both the origin $\mathbf{0} = (0, \ldots, 0)^T$ and $\mathbf{n} = (n_1, \ldots, n_d)^T$ and whose successive elements differ by at most one in each coordinate. In other words, the elements of $D(\mathbf{n})$ correspond to walks from $\mathbf{0}$ to $\mathbf{n}$ in which only positive steps from the d-dimensional unit hypercube are allowed. This follows [12]. The cardinalities $|D(\mathbf{n})|$ are referred to as (generalized) Delannoy numbers. For more about generalizations of the Delannoy numbers, we refer the reader to [2, 12].

When all the n_i share a common value n, we have $\mathbf{n} = (n, \ldots, n)^T$ and we refer to the cardinalities $|D(\mathbf{n})|$ as the (d-dimensional) *central* Delannoy numbers. In this section we use an inclusion/exclusion argument to find an expression for the general Delannoy numbers which specializes to a useful expression for the central Delannoy numbers in Theorem 4.

3.1. **Delannoy paths with k steps.** It is common to refer to the size of a Delannoy path by the number of steps it contains, rather than the number of elements it has as a chain in $L(\mathbf{n})$. In other words, suppose a chain ξ in $D(\mathbf{n})$ has elements

$$\mathbf{a}_0 \prec \mathbf{a}_1 \prec \cdots \prec \mathbf{a}_k.$$

Then we say ξ has k steps. (Notice that ξ has $k+1$ elements, and hence, length $k+1$ as a chain). The set of all k-step Delannoy paths is denoted by $D_k(\mathbf{n})$.

Due to symmetry, the ordering of the dimensions in L is often irrelevant, so we frequently assume that $n_1 \leq n_2 \leq \cdots \leq n_d$. Under this assumption, it is easy to show that the minimum number of steps a Delannoy path can have is n_d, while the maximum is $n_1+\cdots+n_d$, which corresponds to $k_{\max}-1$ from earlier in this article.

Finally, we remark that, since each k-step Delannoy path begins and ends with the points $\mathbf{a}_0 = \mathbf{0}$ and $\mathbf{a}_k = \mathbf{n}$, we could equivalently represent a path $\mathbf{a}_0 \prec \mathbf{a}_1 \prec \cdots \prec \mathbf{a}_k$ by the sequence $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_k$, where each $\mathbf{b}_i = \mathbf{a}_i - \mathbf{a}_{i-1}$. In this representation, the $\mathbf{b}_i$ are nonzero d-tuples of 0s and 1s. This observation is the key to the following result.

Theorem 2. *Fix $\mathbf{n} \in \mathbb{N}^d$ such that $n_1 \leq n_2 \leq \cdots \leq n_d$. Let $k_{\max} = n_1+\cdots+n_d+1$. Then, for each k ($n_d \leq k \leq k_{\max}-1$), the number of k-step Delannoy paths in the lattice $L(\mathbf{n})$ is given by*

$$|D_k(\mathbf{n})| = \binom{k}{n_d} \sum_{i=0}^{k-n_d} (-1)^i \binom{k-n_d}{i} \prod_{j=1}^{d-1} \binom{k-i}{n_j}.$$

Proof. Observe that each k-step Delannoy path $\mathbf{a}_0 \prec \mathbf{a}_1 \prec \cdots \prec \mathbf{a}_k$ corresponds uniquely to a sequence $\mathcal{B} = \langle \mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_k \rangle$, where each $\mathbf{b}_i = \mathbf{a}_i - \mathbf{a}_{i-1}$. Each $\mathbf{b}_i$ is a nonzero d-tuple $(b_{i1}, b_{i2}, \ldots, b_{id})^T$ of 0s and 1s. By the definition of a Delannoy path, projection of $\mathcal{B}$ onto the j-coordinate must give a sequence $\mathcal{B}_j = \langle b_{1j}, b_{2j}, \ldots, b_{kj} \rangle$ of 0s and 1s that contains precisely n_j ones, for each j ($1 \leq j \leq d$).

We count the number of such sequences $\mathcal{B}$ as follows. First, we choose the sequence $\mathcal{B}_d = \langle b_{1d}, b_{2d}, \ldots, b_{kd} \rangle$ of n_d ones and $k-n_d$ zeros. There are $\binom{k}{n_d}$ choices for $\mathcal{B}_d$. Next, we must choose sequences $\mathcal{B}_j$ for $1 \leq j \leq d-1$ in such a way that each sequence has exactly n_j ones, but we must also ensure that the resulting sequence $\mathcal{B}$ has no zero terms. This amounts to ensuring that, for each zero term in $\mathcal{B}_d$, there is at least one $\mathcal{B}_j$ which is nonzero in that term. In other words, for each i where $b_{id} = 0$, there must be at least one j ($1 \leq j \leq d-1$) for which $b_{ij} = 1$. This is achieved by the method of inclusion/exclusion, as follows.

Let $\mathcal{Z} = \{i \,|\, b_{id} = 0\}$, and for each $\mathcal{T} \subseteq \mathcal{Z}$, let $s(\mathcal{T})$ be the number of sequences $\mathcal{S} = \langle \mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_k \rangle$, such that all of the conditions (i)-(iv) hold below.

(i) For each i ($1 \leq i \leq k$), the term $\mathbf{s}_i$ is a d-tuple $(s_{i1}, s_{i2}, \ldots, s_{id})^T$ of 0s and 1s;
(ii) The d-projection of $\mathcal{S}$ satisfies $\mathcal{S}_d = \mathcal{B}_d$.
(iii) Each j-projection $\mathcal{S}_j = \langle s_{1j}, s_{2j}, \ldots, s_{kj} \rangle$ has precisely n_j ones.
(iv) For each $t \in \mathcal{T}$, the term $\mathbf{s}_t = \mathbf{0}$.

To count $s(\mathcal{T})$, we note that condition (ii) fixes the d-projection of $\mathcal{S}$. It remains to satisfy (iii) by forming $d-1$ independent sequences $\mathcal{S}_j$, each with the specified number n_j of ones, and subject to the constraint (iv) that $s_{tj} = 0$ for $t \in \mathcal{T}$. So for each j, there are $\binom{k-t}{n_j}$ choices for such a sequence, where $t = |\mathcal{T}|$. Taken together, we arrive at the total $s(\mathcal{T}) = \prod_{j=1}^{d-1} \binom{k-t}{n_j}$.

Now, since $|\mathcal{T}|$ can range from 0 to $k-n_d$, the number of sequences $\mathcal{B}$ that have the specified d-projection $\mathcal{B}_d$ and contain no zero terms is, by inclusion/exclusion,

$$\sum_{\mathcal{T}\subseteq\mathcal{Z}}(-1)^{|\mathcal{T}|}s(\mathcal{T}) = \sum_{t=0}^{k-n_d}(-1)^t\binom{k-n_d}{t}\prod_{j=1}^{d-1}\binom{k-t}{n_j}.$$

Recalling the number of choices for $\mathcal{B}_d$ was $\binom{k}{n_d}$, we obtain the result. □

3.2. **The total number of Delannoy paths.**

Theorem 3. *Fix $\mathbf{n}\in\mathbb{N}^d$ such that $n_1\le n_2\le\cdots\le n_d$ and let $k'_{\max}=n_1+\cdots+n_{d-1}+1$. Then the total number of Delannoy paths in the lattice $L(\mathbf{n})$ is given by*

$$|D(\mathbf{n})| = \sum_{k=1}^{k'_{\max}}\sum_{i=1}^{k}(-1)^{i+k}\binom{k-1}{i-1}\binom{n_d+k-1}{n_d}\prod_{j=1}^{d-1}\binom{n_d+i-1}{n_j}.$$

Proof. To find $|D(\mathbf{n})|$, we sum the expression for $|D_k(\mathbf{n})|$ from Theorem 2 over all k from n_d to $k_{\max}-1$ to obtain:

$$|D(\mathbf{n})| = \sum_{k=n_d}^{k_{\max}-1}\binom{k}{n_d}\sum_{i=0}^{k-n_d}(-1)^i\binom{k-n_d}{i}\prod_{j=1}^{d-1}\binom{k-i}{n_j}.$$

Reindexing the outer sum, this simplifies to

$$|D(\mathbf{n})| = \sum_{k=1}^{k'_{\max}}\binom{n_d+k-1}{n_d}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}\prod_{j=1}^{d-1}\binom{n_d-1+k-i}{n_j}.$$

Replacing i by $k-i$ reverses the order of the inner sum, and simplification gives the desired result. □

Specializing to the central case, we have the following.

Theorem 4. [9] *Fix $\mathbf{n}\in\mathbb{N}^d$ such that $n_i=n$ for all i $(1\le i\le d)$. Then*

$$C(\mathbf{n}) = 2^{n+1}D(\mathbf{n}).$$

Proof. Immediate from Theorems 1 and 3. □

4. Generating Functions

For dimension $d=2$, the total number of chains $|C(n_1,n_2)|$ satisfies the recurrence

$$|C(n_1,n_2)| = 2\,|C(n_1-1,n_2)| + 2\,|C(n_1,n_2-1)| - 2\,|C(n_1-1,n_2-1)|$$

for $n_1,n_2\ge 1$, and $|C(n_1,n_2)| = 2^{n_1+n_2}$ whenever $n_1n_2=0$.

Similarly, the total number of Delannoy paths $|D(n_1,n_2)|$ satisfies the recurrence

$$|D(n_1,n_2)| = |D(n_1-1,n_2)| + |D(n_1,n_2-1)| + |D(n_1-1,n_2-1)|$$

for $n_1,n_2\ge 1$, and $|D(n_1,n_2)|=1$ whenever $n_1n_2=0$.

Using these recurrences, the generating functions can be derived. Let $g_2^{\mathrm{C}}(x,y)$ and $g_2^{\mathrm{D}}(x,y)$ be the generating functions for $|C(n_1,n_2)|$ and $|D(n_1,n_2)|$, respectively. So

$$g_2^{\mathrm{C}}(x,y) = \frac{2}{1-2x+2y-2xy},$$

and

$$g_2^{\mathrm{D}}(x, y) = \frac{1}{1 - x - y - xy}.$$

Generalizing these results to higher dimensions requires some additional notation. For any positive integer d, let $[d]$ denote the set $\{1, 2, \ldots, d\}$. Also, let $\mathcal{B}_d = \{0, 1\}^d \setminus \{\mathbf{0}\}$. Recall the *support* of $\mathbf{a} \in \mathbb{N}^d$ is $\operatorname{supp}(\mathbf{v}) = \{\, j \in [d] \mid a_j \neq 0 \,\}$. Observe that for $\mathbf{v} \in \mathcal{B}_d$ we have that $\operatorname{wt}(\mathbf{v}) = |\operatorname{supp}(\mathbf{v})|$, which is nonzero by the definition of $\mathcal{B}_d$.

Let $\mathbf{n} \in \mathbb{N}^d$ with $\mathbf{n} = (n_1, n_2, \ldots, n_d)$. If $T = \{\, i_1, i_2, \ldots, i_m \,\}$ is a nonempty subset of $[d]$, then we define $\mathbf{n}_T$ to be the vector of length $d - m$ obtained by removing the $i_1, i_2, \ldots, i_m$ components of $\mathbf{n}$. When $T = \emptyset$ we set $\mathbf{n}_T = \mathbf{n}$.

For arbitrary dimension d it is easily verified that for $\mathbf{n} \in \mathbb{P}^d$ we have

$$|C(\mathbf{n})| = \sum_{\mathbf{v} \in \mathcal{B}_d} 2(-1)^{\operatorname{wt}(\mathbf{v})+1} \, |C(\mathbf{n} - \mathbf{v})| \, ,$$

and

$$|D(\mathbf{n})| = \sum_{\mathbf{v} \in \mathcal{B}_d} |D(\mathbf{n} - \mathbf{v})| \, .$$

Because these recurrences are similar in form, we treat them both as special cases of a more general form to derive the generating functions uniformly. We begin with the following definition.

Definition 2. *Let $\mathfrak{a} = \langle a_0, a_1, a_2, \ldots \rangle$ be a sequence of real numbers. A sequence of functions $\mathcal{F} = \langle F_1, F_2, F_3, \ldots \rangle$, where $F_d : \mathbb{N}^d \to \mathbb{R}$ for all $d \in \mathbb{P}$, is* $\mathfrak{a}$-recurrent *if the following (i)-(iii) hold.*

(i) *For all $d \in \mathbb{P}$ and $\mathbf{n} \in \mathbb{P}^d$,*

$$F_d(\mathbf{n}) = \sum_{\mathbf{v} \in \mathcal{B}_k} a_{\operatorname{wt}(\mathbf{v})} F_d(\mathbf{n} - \mathbf{v}).$$

(ii) *For all $d \in \mathbb{P}$ and nonzero $\mathbf{n} \in \mathbb{N}^d \setminus \mathbb{P}^d$,*

$$F_d(\mathbf{n}) = F_{d-|T|}(\mathbf{n}_T),$$

where $T := [d] \setminus \operatorname{supp}(\mathbf{n})$.

(iii) *For all $d \in \mathbb{P}$,*

$$F_d(\mathbf{0}) = a_0.$$

It is not difficult to see that for a given sequence $\mathfrak{a} = \langle a_0, a_1, a_2, \ldots \rangle$, there is a unique sequence of functions which is $\mathfrak{a}$-recurrent. To illustrate this definition, let $F_d(\mathbf{n}) = |C(\mathbf{n})|$, where $\mathbf{n} \in \mathbb{N}^d$. So the sequence of chain numbers is $\mathfrak{a}$-recurrent with $\mathfrak{a} = \langle 2, 2, -2, 2, -2, 2, \ldots \rangle$, where the signs alternate after the first two entries. Similarly, let $F_d(\mathbf{n}) = |D(\mathbf{n})|$, where $\mathbf{n} \in \mathbb{N}^d$. So the sequence of Delannoy numbers is $\mathfrak{a}$-recurrent with $\mathfrak{a} = \langle 1, 1, 1, \ldots \rangle$.

Theorem 5. *Let $\mathcal{F} = \langle F_1, F_2, F_3, \ldots \rangle$ be an $\mathfrak{a}$-recurrent sequence of functions with $\mathfrak{a} = \langle a_0, a_1, a_2, \ldots \rangle$. For $d \in \mathbb{P}$, let $g_d(\mathbf{x}) = g_d(x_1, x_2, \ldots, x_d)$ be the generating function for F_d. Then*

$$g_d(\mathbf{x}) = a_0 \left(1 - \sum_{\emptyset \neq S \subseteq [d]} a_{|S|} \prod_{i \in S} x_i \right)^{-1}.$$

Proof. We proceed by induction on d. If $d=1$, then $F_1(n)=a_1F_1(n-1)$ for $n\in\mathbb{P}$ by Def. 2(i) and $F_1(0)=a_0$ by Def. 2(iii). So

$$\sum_{n=1}^{\infty}F_1(n)x^n=\sum_{n=1}^{\infty}a_1F_1(n-1)x^n.$$

Thus $g_1(x)-a_0=xa_1g_1(x)$ giving that $g_1(x)=a_0(1-a_1x)^{-1}$, as desired.

Now fix an integer $d\geq 2$ and suppose that the statement holds for all $j\in[d-1]$. Using the fact that our sequence of functions is $\mathfrak{a}$-recurrent, we have that

$$\sum_{\mathbf{n}\in\mathbb{P}^d}F_d(\mathbf{n})x_1^{n_1}x_2^{n_2}\cdots x_d^{n_d}=\sum_{\mathbf{n}\in\mathbb{P}^d}\sum_{\mathbf{v}\in\mathcal{B}_d}a_{\mathrm{wt}(\mathbf{v})}F_d(\mathbf{n}-\mathbf{v})x_1^{n_1}x_2^{n_2}\cdots x_d^{n_d}. \tag{7}$$

Consider the left-hand side of (7). By inclusion-exclusion and the definition of $\mathfrak{a}$-recurrent, we have

$$\sum_{\mathbf{n}\in\mathbb{P}^d}F_d(\mathbf{n})x_1^{n_1}x_2^{n_2}\cdots x_d^{n_d}=(-1)^da_0+\sum_{S\subsetneq[d]}(-1)^{|S|}g_{d-|S|}(\mathbf{x}_S). \tag{8}$$

Now consider the right-hand side of (7). Let $\mathbf{v}\in\mathcal{B}_d$ and let $S_{\mathbf{v}}:=\mathrm{supp}(\mathbf{v})=\{\,i_1,i_2,\ldots,i_{\mathrm{wt}(\mathbf{v})}\,\}$. Note that $\mathrm{wt}(\mathbf{v})\geq 1$. Again by inclusion-exclusion, for this particular $\mathbf{v}$, we have

$$\sum_{\mathbf{n}\in\mathbb{P}^d}a_{\mathrm{wt}(\mathbf{v})}F_d(\mathbf{n}-\mathbf{v})x_1^{n_1}x_2^{n_2}\cdots x_d^{n_d}= a_{\mathrm{wt}(\mathbf{v})}x_{i_1}x_{i_2}\cdots x_{i_{\mathrm{wt}(\mathbf{v})}}\sum_{T\subseteq[d]\setminus S_{\mathbf{v}}}(-1)^{|T|}g_{d-|T|}(\mathbf{x}_T) \tag{9}$$

By (7), the expression on the right side of (8) must equal the sum over all $\mathbf{v}\in\mathcal{B}_d$ of the expression on the right side of (9), giving

$$(-1)^da_0+\sum_{S\subsetneq[d]}(-1)^{|S|}g_{d-|S|}(\mathbf{x}_S)= \sum_{\mathbf{v}\in\mathcal{B}_d}a_{\mathrm{wt}(\mathbf{v})}x_{i_1}x_{i_2}\cdots x_{i_{\mathrm{wt}(\mathbf{v})}}\sum_{T\subseteq[d]\setminus S_{\mathbf{v}}}(-1)^{|T|}g_{d-|T|}(\mathbf{x}_T) \tag{10}$$

Each $\mathbf{v}\in\mathcal{B}_d$ corresponds to a unique nonempty subset $V\subseteq[d]$ and conversely, so (10) can be rewritten as

$$(-1)^da_0+\sum_{S\subsetneq[d]}(-1)^{|S|}g_{d-|S|}(\mathbf{x}_S)=\sum_{\emptyset\neq V\subseteq[d]}a_{|V|}\prod_{i\in V}x_i\sum_{T\subseteq[d]\setminus V}(-1)^{|T|}g_{d-|T|}(\mathbf{x}_T). \tag{11}$$

Swapping the order of summation on the right yields

$$(-1)^da_0+\sum_{S\subsetneq[d]}(-1)^{|S|}g_{d-|S|}(\mathbf{x}_S)=\sum_{T\subsetneq[d]}(-1)^{|T|}g_{d-|T|}(\mathbf{x}_T)\sum_{\emptyset\neq V\subseteq[d]\setminus T}a_{|V|}\prod_{i\in V}x_i. \tag{12}$$

Collecting all instances of $g_d(\mathbf{x})$ on the left side, we obtain

$$g_d(\mathbf{x})\left(1-\sum_{\emptyset\neq V\subseteq[d]} a_{|V|}\prod_{i\in V} x_i\right) = (-1)^{d+1}a_0 - \sum_{\emptyset\neq S\subsetneq[d]} (-1)^{|S|} g_{d-|S|}(\mathbf{x}_S) + \sum_{\emptyset\neq T\subsetneq[d]} (-1)^{|T|} g_{d-|T|}(\mathbf{x}_T) \sum_{\emptyset\neq V\subseteq[d]\setminus T} a_{|V|}\prod_{i\in V} x_i. \tag{13}$$

It remains to show that the right side equals a_0. To this end, observe that the two sums on the right can be combined. Doing so reduces the right side of the above to

$$(-1)^{d+1}a_0 - \sum_{\emptyset\neq S\subsetneq[d]} (-1)^{|S|} g_{d-|S|}(\mathbf{x}_S)\left(1-\sum_{\emptyset\neq V\subseteq[d]\setminus S} a_{|V|}\prod_{i\in V} x_i\right).$$

By the induction hypothesis,

$$g_{d-|S|}(\mathbf{x}_S) = a_0\left(1-\sum_{\emptyset\neq V\subseteq[d]\setminus S} a_{|V|}\prod_{i\in V} x_i\right)^{-1}.$$

Therefore, substitution into (13) gives us

$$g_d(\mathbf{x})\left(1-\sum_{\emptyset\neq V\subseteq[d]} a_{|V|}\prod_{i\in V} x_i\right) = (-1)^{d+1}a_0 - \sum_{\emptyset\neq S\subsetneq[d]} (-1)^{|S|}a_0. \tag{14}$$

$$= a_0\left[(-1)^{d+1} - \sum_{i=1}^{d-1}\binom{d}{i}(-1)^i\right]. \tag{15}$$

The bracketed expression equals 1, by a well-known identity [1, Th. 1.7], and the result now follows. □

Corollary 4. *Let $g_3^{\mathrm{D}}(x,y,z)$ be the generating function for the 3-dimensional Delannoy numbers. Then*

$$g_3^{\mathrm{D}}(x,y,z) = \frac{1}{1-x-y-z-xy-xz-yz-xyz}.$$

Corollary 5. *Let $g_3^{\mathrm{C}}(x,y,z)$ be the generating function for the 3-dimensional chain numbers. Then*

$$g_3^{\mathrm{C}}(x,y,z) = \frac{2}{1-2(x+y+z-xy-xz-yz+xyz)}.$$

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