The 1-Relaxed Edge-Sum Labeling Game

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1-Relaxed Modular Edge-sum Labeling

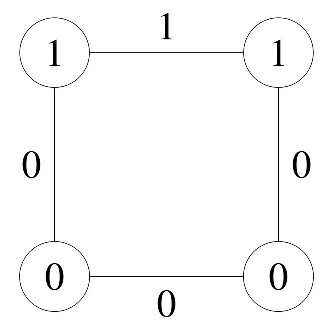
Let $G = (V, E)$ be a simple graph with no isolated vertices, and let $n \in \mathbb{N}$. For each $v \in V$, let $N'(v)$ be the set of edges at v. We define a ***labeling*** of G using elements of $\mathbb{Z}_n$, called a ***modular edge-sum labeling***, in the following way:

- Assign a label from $\mathbb{Z}_n$ to each edge $e \in E$, denoted $w(e)$.
- For all $v \in V$, define the label of v in the following manner:

$$\ell(v) = \sum_{e \in N'(v)} w(e),$$

where the summation is computed modulo n.

A ***1-relaxed edge-sum labeling*** of graph G is for each labeled vertex v in G, v has at most one neighbor vertex that has the same label.



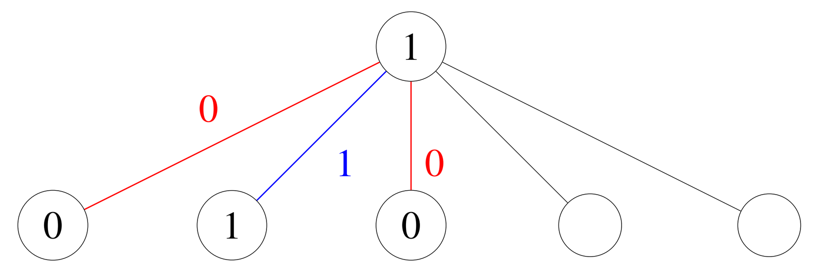
1-relaxed edge-sum labeling in $\mathbb{Z}_2$

1-Relaxed Edge-sum Labeling Game

The players, Alice and Bob, alternate turns with Alice going first. For each turn of the game, the player chooses an unlabeled edge e, and assigns to e a label $w(e) \in \mathbb{Z}_n$. At any point in the game, let L' be the set of labeled edges. For any turn t, let v be a vertex for which $N'(v) \cap L' \neq \emptyset$. We define the label for v at this point in the game to be $\ell_t(v)$ according to the following:

$$\ell_t(v) = \sum_{e \in N'(v) \cap L'} w(e),$$

where this sum is computed modulo n. If a vertex is not incident with any labeled edges, that vertex remains unlabeled.
Let L be the set of labeled vertices. We call a 1-relaxed edge-sum labeling ***legal*** if, for each $v \in L$, we have $\mathrm{def}(v) \leq 1$, and we require that the players maintain a legal 1-relaxed edge-sum labeling at each stage of the game.
Alice wins the game if play continues until $L' = E(G)$; otherwise, Bob wins. The least n such that Alice has a winning strategy for this game on G is called the *1-relaxed edge-sum labeling game number of* G, denoted ${}^1\Lambda_g(G)$.



Alice wins in this graph $K_{1,k}$ in $\mathbb{Z}_2$.(Alice plays red, Bob plays blue)
${}^1\Lambda_g(K_{1,k}) = 2$

Main Results

Theorems

Theorem 1. *For any path P with $|V(P)| \geq 3$, ${}^1\Lambda(P) = 2$.*

Theorem 2. *For any path P, ${}^1\Lambda_g(P) \leq 3$.*

Theorem 3. *For any tree T, ${}^1\Lambda_g(T) \leq \Delta(T) + 2$.*

Theorem 4. *There exists a tree for which ${}^1\Lambda_g(T) \geq 3$.*

Theorem 5. *For any tree T with $|V(T)| \geq 3$, ${}^1\Lambda(T) = 2$.*

Theorem 6. *For any tree T, ${}^1\Lambda_g(T) \leq \boldsymbol{\Delta}(T) + 2$*

Theorem 7. *Let G be a complete bipartite graph with partite sets A and B such that $|A|, |B| \geq 2$. If $|A|$ or $|B|$ is even, then ${}^1\Lambda(G) = 2$. Otherwise, ${}^1\Lambda(G) = 3$.*

Lemmas

Lema 1. *At any point during the game, any unlabeled edge between two previously labeled vertices can be legally assigned the label 0.*

Lema 2. *For any connected graph G,*

- *If $|V(G)| = 2$, then ${}^1\Lambda(G) = 1$.*
- *If $|V(G)| \geq 3$, then ${}^1\Lambda(G) \geq 2$.*

Lema 3. *If Alice follows the Activation Strategy when playing the 1-relaxed edge-sum labeling game on a tree T, then at any point in the game, any unlabeled vertex u will have at most two active children.*

Lema 4. *Suppose Alice and Bob are playing the 1-relaxed edge-sum labeling game on a tree which is rooted at any vertex r, Alice is employing the Activation Strategy, and has chosen to label e on turn $t > 1$. Then the captive end of e is not in $R(r)$.*

Lema 5. *Let P be a maximum path in a tree T with an end vertex u. Then the unique neighbor, v, of u in T has at most one non-leaf neighbor in T.*

Lema 6. *Let G be a bipartite graph with partite sets A and B, and consider a 1-relaxed edge-sum labeling in $\mathbb{Z}_n$. Let L be the set of labeled vertices. For any t,*

$$\sum_{v \in A} \ell(v) = \sum_{u \in B} \ell(u),$$

where the sums are computed modulo n.

Lema 7. *Let G be a complete bipartite graph with partite sets A and B, and let $\ell : V(G) \to \mathbb{Z}_n$ be a 1-relaxed 1-relaxed edge-sum labeling of G. If there exist $x, y \in A$ such that $\ell(x) = \ell(y) = a$, then there can exist no $z \in B$ such that $\ell(z) = a$.*

The Activation Strategy

We now present an Activation Strategy derived from one developed in [6] for Alice to use when playing the 1-relaxed edge-sum labeling game on trees.
Assume Alice and Bob are playing the 1-relaxed edge-sum labeling game on a tree T which is rooted at a vertex r. For any vertex $x \in V(T)$, define the *parent* of x (denoted $p(x)$) to be the neighbor of x such that $p(x)$ is on the path from x to r. Also, if $y = p(x)$, we say that x is a *child* of y.
At any point in the game, let L be the set of labeled vertices, L' be the set of labeled edges, U be the set of unlabeled vertices, and U' be the set of unlabeled edges. Alice will maintain a set A of *active* vertices which will begin empty, then may grow throughout the game.
Alice will begin by assigning a label to an edge between r and a neighboring leaf, if one exists. Otherwise, Alice will assign a label to any edge at r. Now, assume Bob assigns a label to any edge in $E(T)$. Let b be the end of this edge such that the distance $d(b, r)$ is minimized. Alice will choose and label an edge e in the following manner.

- Initial Step If $p(b) \in U$, then set $x := p(b)$ and go to the Recursive Step.
  Else, let u be any unlabeled vertex such that $p(u) \in L$, set $A := A \cup \{u\}$, and go to the Edge-Choice Step.
- Recursive Step If $x \notin A$ and $p(x) \in U$, then set $A := A \cup \{x\}$, set $x := p(x)$, and repeat the Recursive Step.
  Else, set $A := A \cup \{x\}$, set $u := x$, and go to the Edge-Choice Step.
- Edge-Choice Step
  If u is a leaf, let $v := p(u)$, set $A := A \cup \{v\}$, let $e = uv$, and go to the Labeling Step.
  Else, if u has an active child, let v be an active child of u, let $e = uv$, and go to the Labeling Step.
  Else, let v be any child of u, set $A := A \cup \{v\}$, let $e = uv$, and go to the Labeling Step.
- Labeling Step From the set of labels which minimize the defect of u, choose the label which minimizes the defect of v, and assign it to e.

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