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The Relaxed Game Chromatic Index of *k*-Degenerate Graphs

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THE RELAXED GAME CHROMATIC INDEX OF k-DEGENERATE GRAPHS

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ABSTRACT. The (r, d)-relaxed coloring game is a two-player game played on the vertex set of a graph G. We consider a natural analogue to this game on the edge set of G called the (r, d)-relaxed edge-coloring game. We consider this game on trees and more generally, on k-degenerate graphs. We show that if G is k-degenerate with $\Delta(G) = \Delta$, then the first player, Alice, has a winning strategy for this game with $r = \Delta + k - 1$ and $d \geq 2k^2 + 4k$.

1. INTRODUCTION

We explore the connections between three well studied areas of graph coloring: game coloring, edge coloring, and defect coloring. Game coloring was first introduced by Bodlaender in 1991 [1]. In the usual formulation of the game, two players, Alice and Bob, alternate coloring the uncolored vertices of a graph G from a set of r colors X. An uncolored vertex v may be colored $\alpha \in X$ if v has no neighbors already colored α. Alice wins the game if all vertices are eventually colored; otherwise, Bob wins. In this case, there is an uncolored vertex for which there is no allowable color. The least r such that Alice has a winning strategy for this game is called the *game chromatic number of* G, denoted $\chi_g(G)$. This parameter has been studied extensively in a number of papers [2, 3, 4, 5, 6, 7, 8, 9].

Chou, Wang, and Zhu [10] introduced a variation of the above game based on the idea of relaxed colorings (or defect colorings). This variation is called the (r, d)*-relaxed coloring game*. The parameters r and d are the number of colors and the defect, respectively, with r a positive integer and d a nonnegative integer. This game differs from the original game in rules for when a vertex can be colored with a particular color. In the original game, a color is allowed for a vertex if at every step in the game, all of the color classes induce independent sets. In this variation, the color classes must induce subgraphs with maximum degree at most d. Hence, adjacent vertices may receive the same color, but no vertex can have more than d neighbors colored with its own color. For a fixed d, the least r such that Alice has a winning strategy for this game is called the *d-relaxed game chromatic*

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number of G, denoted ${}^{d}\chi_{\mathrm{g}}(G)$. Given that there are two initial parameters defining the game (r and d), we can also fix r and consider the least d for which Alice has a winning strategy. The parameter is called the *r-game defect of* G, denoted $\mathrm{def}_{\mathrm{g}}(G, r)$. This variation of the game has since been studied in number of papers [11, 12, 13, 14]. These papers have consider this game with trees, outerplanar graphs, planar graphs, partial k-trees, and more technical superclasses. In addition, defect colorings (without the game concept) have been studied in many papers [15, 16, 17, 18, 19].

We introduce a natural variation of the (r, d)-relaxed coloring game which is played on the edge set of a graph G rather than the vertex set of G. Of course, this could then be seen as playing the (r, d)-relaxed coloring game on the line graph of G. This variation is called the *(r, d)-relaxed edge-coloring game*. Suppose Alice and Bob are playing this game on a graph G with a set of colors X. An uncolored edge e may be colored $\alpha \in X$ if two conditions hold: there are at most d edges incident with e that have previously been colored α, and, if e' is incident with e and e' has already been colored α, then there are at most $d-1$ edges incident with e' that have previously been colored α. For a fixed d, we define the *d-relaxed game chromatic index of* G, denoted ${}^{d}\chi'_{\mathrm{g}}(G)$, to be the least r such that Alice has a winning strategy for this game. For a fixed r, we define the *r-edge-game defect of* G, denoted $\mathrm{def}'_{\mathrm{g}}(G, r)$ to be the least d for which Alice has a winning strategy.

We note the following unexpected feature of the (r, d)-relaxed coloring game and (r, d)-relaxed edge-coloring game. While we might expect that if ${}^{d}\chi_{\mathrm{g}}(G) = k$, then ${}^{d'}\chi_{\mathrm{g}}(G) \leq k$ for all $d' > d$, this is not always the case. For example, if $G = K_{n,n}$, then ${}^{0}\chi_{\mathrm{g}}(G) = 3$ but ${}^{1}\chi_{\mathrm{g}}(G) = n$. Thus, if $n \geq 4$, we have that ${}^{1}\chi_{\mathrm{g}}(G) > {}^{0}\chi_{\mathrm{g}}(G)$.

The strategies we employ are activation strategies, as used by Kierstead [6]. This strategy can be seen as a culmination of the work in many papers [4, 20, 3, 8, 9]. An important aspect of this strategy is the notion of a linear ordering of the vertices of a graph. We now define some of the terminology and notation that will be important.

Let $G = (V, E)$ be a finite simple graph. Let $\Pi(G)$ be the set of all linear orderings of V. Consider a linear ordering $L \in \Pi(G)$ with $L = v_1 v_2 \ldots v_n$. For any v_i we denote the neighborhood of v_i in G by $N_G(v_i)$. Using L we now define the following:

$$
\begin{aligned}
V^{+}_{G,L}(v_i) &= \{\, v_j \mid j < i \,\} & V^{-}_{G,L}(v_i) &= \{\, v_j \mid j > i \,\} \\
N^{+}_{G,L}(v) &= V^{+}_{G,L}(v) \cap N_G(v) & N^{-}_{G,L}(v) &= V^{-}_{G,L}(v) \cap N_G(v) \\
V^{+}_{G,L}[v] &= V^{+}_{G,L}(v) \cup \{v\} & V^{-}_{G,L}[v] &= V^{-}_{G,L}(v) \cup \{v\} \\
N^{+}_{G,L}[v] &= N^{+}_{G,L}(v) \cup \{v\} & N^{-}_{G,L}[v] &= N^{-}_{G,L}(v) \cup \{v\}
\end{aligned}
$$

As the notation suggests, it is sometimes convenient to think of the edges of G being oriented with respect to L. The edge $e = v_i v_j \in E$ is oriented

$v_i \to v_j$ if and only if $i > j$. We denote the degrees of a vertex by

$$d_G(v) = |N_G(v)|,$$

$$d^+_{G,L}(v) = \left|N^+_{G,L}(v)\right|,$$

and

$$d^-_{G,L}(v) = \left|N^-_{G,L}(v)\right|.$$

Finally, we let $\Delta^+_L(G) = \max_{v\in V} d^+_{G,L}(v)$. We call $\Delta^+_L(G)$ the *maximum backdegree of G*. When the graph G is clear from the context, we drop the subscript G from all notation defined above. Similarly, when the linear ordering L is clear from the context, we drop the subscript L. Consider a subset $V' \subseteq V$. For ease of notation, we will write $u = \min_L V'$ to mean that u is the vertex of least index in V' relative to L. In addition, we refer to u as the L-least vertex in V', or as simply the least vertex in V'. Finally, v_1 is the least vertex in L.

2. Strategy for Trees

The following result was proven by Lam, Shiu, and Xu [21] and Cai and Zhu [2].

Theorem 1. *Let T be a tree with $\Delta(T) = \Delta$. Then $\chi'_g(T) \leq \Delta + 2$.*

The result of Faigle, et al., [4] stating that $\chi_{\mathrm{g}}(T) \leq 4$ for all trees T raised questions about relaxed game coloring with 2 and 3 colors on trees. We ask the following analogous questions:

Question 1. *Is it true that there exists a defect d such that ${}^d\chi'_g(T) \leq \Delta(T) + 1$ for every tree T?*

Question 2. *Is it true that there exists a defect d such that ${}^d\chi'_g(T) \leq \Delta(T)$ for every tree T?*

We answer both of these questions in the affirmative, providing Alice with a winning strategy that is similar to that used by Dunn and Kierstead [11], coloring edges rather than vertices.

Let $T = (V, E)$ be a tree and let X be a set of colors. Pick a root r and direct all of the edges in T toward r. For each $v \in V - \{r\}$, define $p(v)$ to be the unique outneighbor of v. With this definition, every edge $e \in E$ can be expressed in the form $xp(x)$ for some $x \in V$. We define the set

$$E_0 = \{\, e \in E \mid e = xp(x) \text{ for some } x \text{ with } p(x) = r \,\}.$$

Let $e \in E - E_0$ with $e = xp(x)$ for some x. We define $p(e)$ to be the edge $p(x)p^2(x)$ where $p^2(x) = p(p(x))$. Notice that $p(e)$ is uniquely defined since $p(x)$ is uniquely defined. We call $p(e)$ the *parent* of e and e a *child* of $p(e)$.

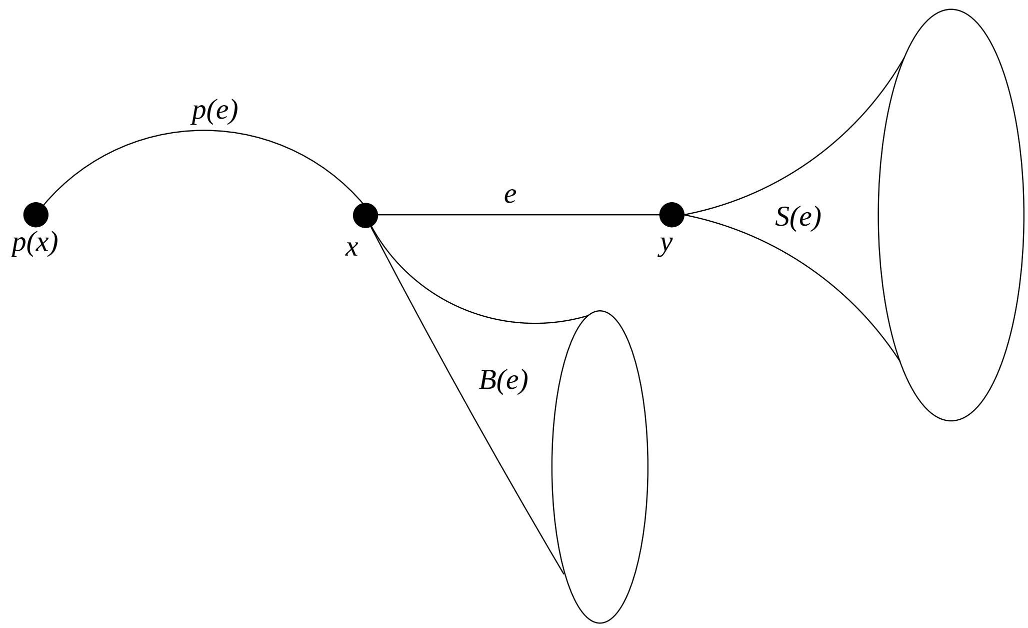

Figure 1. $B(e)$, $S(e)$, $p(x)$, and $p(e)$, relative to x, e, and L.

If $e \in E_0$, we simply set $\{p(e)\} = \emptyset$. For any edge $e \in E$ where $e = xp(x)$, we define the following sets:

$$\begin{aligned} B(e) &= \{\, yp(y) \in E \mid p(y) = p(x) \,\} \\ R(e) &= B(e) \cup \{p(e)\} \\ S(e) &= \{\, wp(w) \in E \mid p(w) = x \,\} \end{aligned}$$

Notice that the edges in $S(e)$ are exactly all of the children of edge e. We denote the set of all edges incident to e by $N(e)$. Observe that $N(e) = R(e) \cup S(e)$.

As with $p^2(x)$ above, we define $p^2(e) = p(p(e))$, if such an edge exists. Inductively, if $p^i(e)$ is defined, we define $p^{i+1}(e) = p(p^i(e))$. Finally, we define

$$G(e) = \{\, e' \in E \mid e = p^k(e') \text{ for some } k \,\}.$$

We call the edges in $G(e)$ the *descendants* of e.

At any point in the game, let C be the set of colored edges and U be the set of uncolored edges. Once an edge is colored, let $c(e)$ be that color. We define the set $D(e)$ of a colored edge e to be the set of all edges incident with e that have been colored $c(e)$. If $e \in U$ then $D(e) = \emptyset$. So the *defect* of an edge e is defined by $\operatorname{def}(e) = |D(e)|$. We say that a color $\alpha \in X$ is *eligible* for an uncolored edge e if no edge in $R(e)$ has been colored with α. We denote the set of eligible colors for e by $X(e)$. For her strategy, Alice also maintains a set A of *active* edges. This set has the property that every colored edge is active, and once a edge is in A it remains in A for the remainder of the game. When an edge is put into A we say that the edge has been *activated*.

We are now ready to state the strategy that Alice will employ for the (r, d)-relaxed edge-coloring game on trees.

TREE STRATEGY

Alice begins by activating and coloring any edge in E_0 with any color. Suppose Bob has just colored the edge b. Alice's strategy then has two stages: she first searches for the edge e that she will color; she then chooses a color α for that edge.

Search Stage

Initial Step

- If $b \notin E_0$ and $p(b) \in U$, then set $g := p(b)$ and move to the recursive step.
- If $b \notin E_0$, $p(b) \notin E_0$, $p(b) \in C$ with $c(p(b)) = c(b)$, and $p^2(b) \in U$, then set $e := p^2(e)$ and move to the coloring stage.
- Otherwise, let e be any edge whose parent is colored and move to the coloring stage, activating e if e is inactive.

Recursive Step

- If $g \notin A \cup E_0$ and $p(g) \in U$, then activate g, set $g := p(g)$, and repeat the recursive step.
- Otherwise, activate g if inactive, set $e := g$, and move to the coloring stage.

Coloring Stage

- Choose an eligible color for e which minimizes $\mathrm{def}(e)$.

We make an important observation which will help in determining the appropriate cardinalities of X with which Alice can win the game.

Proposition 1. *For any edge $e \in E$, we have that $|R(e)| \leq \Delta(T) - 1$.*

This follows from the definition of $R(e)$. In order for Alice to be successful with this strategy, every edge should have at least one eligible color. We know from Theorem 1 that Alice can win with $d = 0$ and $|X| = \Delta(T) + 2$. So we shall examine this game with $|X| = \Delta(T) + 1$ and $|X| = \Delta(T)$.

Theorem 2. *Let T be a tree with $\Delta(T) = \Delta$. Then $\mathrm{def}'_g(T, \Delta + 1) \leq 1$. Moreover, if $d \geq 1$, then ${}^{d}\chi'_g(T) \leq \Delta + 1$.*

Proof. Let X be a set of colors with $|X| = \Delta + 1$. Alice will use the TREE STRATEGY. We first show that at any point in the game, an uncolored edge

e is incident with at most $\Delta + 1$ active edges. Further, if f is the first child of e to be activated, then the first time an edge in $G(e) - G(f)$ is activated, Alice will color e on this turn.

Let $e = xp(x)$ be an uncolored edge. At $p(x)$, the edge e is incident to at most $\Delta - 1$ active edges. It then suffices to show that e has at most two active children. Let f be the first child of e to be activated. When f is activated, Alice will activate e. When the first edge in $G(e) - G(f)$ is activated, Alice will reach e in the recursive step of her strategy and color e on that turn. Thus e is incident with at most $\Delta + 1$ active edges.

Suppose now that Alice is choosing a color for e. If e is incident with at most Δ distinctly colored edges, there is at least color not used on edges in $N(e)$ which she may choose for e which will not affect the defect of any edge incident with e and will result in $\text{def}(e) = 0$. Otherwise, we must have that e has two distinctly colored children, say f and g, where f was colored first. By the argument above, since e remains uncolored, it must be the case that g was the first vertex colored in $G(e) - G(f)$. Thus, at this time in the game, $\text{def}(g) = 0$. Therefore, Alice can choose to color e with $c(g)$, resulting in $\text{def}(e) = \text{def}(g) = 1$, which affects the defect of no other edges. Note that Bob can also use this strategy at any time. In addition, if $d > 1$, then any edges which eventually achieve higher defect than 1 must have reached that point because of the actions of Bob. This is clear since the above argument holds for any uncolored edge e. Thus every edge will eventually be colored and Alice will win the game. ∎

Theorem 3. *Let T be a tree with $\Delta(T) = \Delta$. Then $\text{def}'_g(T, \Delta) \leq 3$. Moreover, if $d \geq 3$, then ${}^d\chi'_g(T) \leq \Delta$.*

Proof. Let X be a set of colors with $|X| = \Delta$. Again, Alice uses the Tree Strategy. We proceed by proving two facts; one for each part of the definition of an allowable color for an uncolored vertex.

Let e be an uncolored edge and let f and g be the first and second children of e to be activated, respectively. On the turn that f is activated, Alice will activate e. If e remains uncolored on the turn that g becomes active, then Alice will respond by coloring e on that turn. Thus, any uncolored edge has at most two active children. Moreover, if an uncolored edge e has two active children, then Alice will immediately color e on that turn.

Now suppose that $e \in U$ and $f \in S(e) \cup C$. We show that f has at most one child colored with $c(f)$. Suppose not. Let g and h be the first two children of f to be colored such that $c(g) = c(h) = c(f)$. Without loss of generality, we will assume that g is colored before h. We will consider three cases.

Case 1: f is colored before g.

Since $p(g) = p(h) = f$, it must be that Bob colored both g and h, as Alice will always choose to use an eligible color for an edge. Thus, when Bob colored g, Alice was in the initial step of the search stage and chose to immediately color e.

Case 2: f is colored after g and before h.

When g is activated, Alice activates f and takes action at e (e may have been activated previously). Thus, if Bob colors f, Alice colors e before h is colored. So suppose that Alice colors f. When Alice colors f, f is incident with at most $|B(f)|+1 \leq \Delta-1$ colored edges. Thus, Alice would not choose to color f with $c(g)$.

Case 3: f is colored after h.

As in Case 2, when g is activated, Alice activates f and takes action at e. Clearly, Alice did not color h since $h \in B(g)$ implying that $c(g) \notin X(h)$. So when Bob colors h and Alice moves to color f, f is incident with at most $|B(f)| + 2 \leq \Delta$ colored edges. However, two of these edges, g and h, are colored with the same color. Thus, Alice would not choose to color f with $c(g)$.

Thus, f has at most one child colored with $c(f)$, as desired.

Now suppose that Alice has chosen to color e with α. Since $\alpha \in X(e)$, coloring e does not affect the defect of any edge in $R(e)$. By the first argument above, when e is colored, $\operatorname{def}(e) \leq 2$. Suppose that $f \in S(e)$ with $c(f) = \alpha$. By the second argument above, we know that $|D(f) \cup S(f)| \leq 1$. Since $B(f) \subset S(e)$, we have that $|D(f) \cup B(f)| \leq 1$. Thus, once e is colored, we have $\operatorname{def}(f) \leq 3$, as desired.

As in the proof of Theorem 2, we conclude by noting that Bob may borrow this strategy at any time. In addition, if $d > 3$ and an edge e eventually has defect at least d, then it must be through the actions of Bob that this has occurred. At the time an edge e is uncolored, the above arguments show that it is possible to color e with an eligible color α such that coloring e did not increase the defect of any edge e' where $\operatorname{def}(e') > 3$. Therefore, every edge will eventually be colored and Alice will win the game. ■

3. Strategy for k-Degenerate Graphs

We will now give a more general strategy using the techniques developed for trees. Our main goal is to prove a result for k-degenerate graphs, with trees examples of 1-degenerate graphs. Recall that a graph $G = (V, E)$ is k-degenerate if there exists a linear ordering $L = v_1, v_2, \ldots, v_n$ of V such that for every $i \in [n]$, we have that

$$|\{\, j \mid v_i \leftrightarrow v_j \text{ and } j < i \,\}| \leq k.$$

In other words, a graph G is k-degenerate if there is a linear ordering of the vertices of G such that every vertex is adjacent to at most k vertices earlier in the list. For example, trees and forests are 1-degenerate, outerplanar graphs are 2-degenerate, planar graphs are 5-degenerate, and partial k-trees are k-degenerate. Cai and Zhu proved the following theorem about k-degenerate graphs and game coloring [2].

Theorem 4. *Let $G = (V, E)$ be a graph with $\Delta(G) = \Delta$ and suppose that G is k-degenerate. Let H be the line graph of G. Then $\chi_g(H) \leq \Delta + 3k - 1$.*

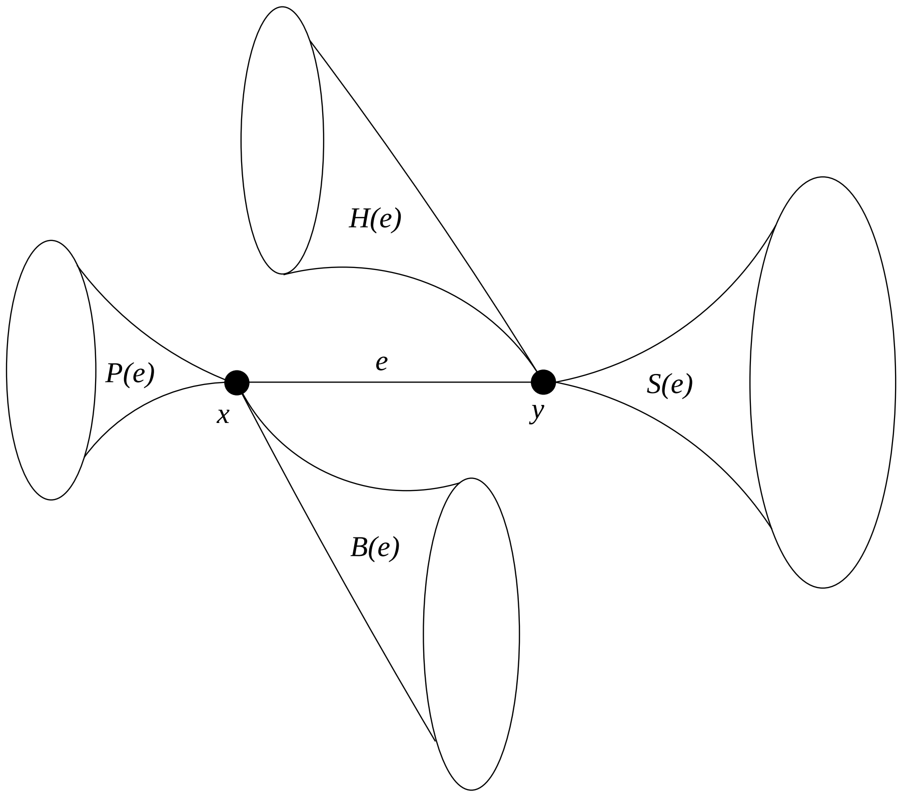

FIGURE 2. The sets $P(e)$, $H(e)$, $B(e)$, and $S(e)$, relative to e and L.

As a consequence, for all k-degenerate graphs G, ${}^{d}\chi'_{\mathrm{g}}(G) \leq \Delta + 3k - 1$ when $d = 0$. Our goal is to reduce the number of colors by finding an allowable defect. We first provide the necessary notation.

Let $G = (V, E)$ be a finite graph and let L be a linear ordering of V. Once L is established, when we write $xy \in E$ we assume that $x < y$ in L. Let $e = xy$ be an edge in G. We now define sets of edges relative to e using L.

$$\begin{aligned} P(e) &= \{\, wx \in E \mid w \in N^{+}(x) \,\} & P[e] &= P(e) \cup \{e\} \\ B(e) &= \{\, xv \in E \mid v \in N^{-}(x) \,\} & B[e] &= B(e) \cup \{e\} \\ H(e) &= \{\, uy \in E \mid u \in N^{+}(y) \,\} & H[e] &= H(e) \cup \{e\} \\ S(e) &= \{\, yz \in E \mid z \in N^{-}(y) \,\} & S[e] &= S(e) \cup \{e\} \\ R(e) &= P(e) \cup B(e) \cup H(e) \end{aligned}$$

As before we will call the vertices in $P(e)$ the *parents* of e and the vertices in $S(e)$ the *children* of e. We again note that $N(e) = R(e) \cup S(e)$.

Consider the linear ordering $\overline{L}$ of E induced lexicographically from L. So $xy < wz$ in $\overline{L}$ if and only if either $x < w$ in L or both $x = w$ and $y < z$ in

L. Although $\overline{L}$ is used in the strategy below, since $\overline{L}$ is determined by L, it is L that determines the strategy.

As with the game on trees, at any time in the game we define U to be the set of uncolored edges and C to be the set of colored edges. For $e \in C$, let the color assigned to e be $c(e)$. Define the *defect set* of e by $D(e) = \{\, e' \in N(e) \mid c(e') = c(e) \,\}$. If $e \in U$, then $D(e) = \emptyset$. In either case, the *defect* of e is defined by $\operatorname{def}(e) = |D(e)|$.

Let $F \subseteq E$. We define

$$\overline{X}(F) = \bigcup_{e' \in F} \{c(e')\},$$

where $\{c(e')\} = \emptyset$ if $e' \in U$. For an edge e, we then let $X(e) = X - \overline{X}(R(e))$. In other words, $X(e)$ is the set of colors not used on any edge in $R(e)$. As before, we call $X(e)$ the set of *eligible colors* for e.

Let $e \in E$. We define $M(e) = U \cap P[e]$. If $M(e) \neq \emptyset$, we define the *mother* of e by $m(e) = \min_{\overline{L}} M(e)$. Note that if $e \in U$ then $m(e)$ must be defined since e itself is a candidate. We define $D^*(e) \subseteq D(e)$ by

$$D^*(e) = \{\, e' \in D(e) \mid m(e') \text{ exists} \,\}.$$

Using this definition, let $F(e) = P(e) \cap (U \cup D^*(e))$. If $F(e) \neq \emptyset$, define the *father* of e by $f(e) = \min_{\overline{L}} F(e)$.

We are now ready to define the strategy that Alice will employ in the (r, d)-relaxed edge-coloring game on k-degenerate graphs.

K STRATEGY

Let $G = (V, E)$ be a graph with linear ordering L of V. Let X be a set of colors. Alice will again maintain a set A, although in this case, it will be a set of edges. Similar to the case with trees, we require that Alice color only edges which have previously been activated. If Bob colors an inactive edge, we will assume that he simultaneously activates it. Alice starts by activating and coloring the least edge in $\overline{L}$. Suppose that Bob has just colored edge b.

Search Stage

Initial Step

- If $f(b)$ exists and $f(b) \in U$, then set $g := f(b)$ and move to the recursive step.
- If $f(b)$ exists and $f(b) \in C$, then set $g := m(f(b))$ and move to the recursive step.
- Otherwise, set $e := \min_{\overline{L}} U$. If e is inactive, activate it. Move to the coloring stage.

Recursive Step

- If $g \notin A$, then activate g, set $g := m(g)$, and repeat recursive step.

- Otherwise, set $e := g$ and move to the coloring stage.

Coloring Stage

- Choose an eligible color for e which minimizes $\operatorname{def}(e)$.

We will refer to moving from e to $f(e)$ in the initial step and e to $m(e)$ in the recursive step as *jumping*. We refer to moving from e to $m(f(e))$ in the initial step as *skipping*. When Alice either activates or colors an edge e, we say that she is *taking action* at e. Note that Alice can take action at a vertex at most twice. This will play an important role in the proof of our main result, which we are now ready to state and prove.

Theorem 5. *Let G be k-degenerate with $\Delta(G) = \Delta$. Then $\operatorname{def}'_g(G, \Delta + k - 1) \leq 2k^2 + 4k$. Moreover, if $d \geq 2k^2 + 4k$, then ${}^d\chi'_g(G) \leq \Delta + k - 1$.*

Proof. Let G be k-degenerate with $\Delta(G) = \Delta$, and let X be a set of colors with $|X| = \Delta + k - 1$. Alice will use the K Strategy. We first note that for any edge e, we have that $|P(e) \cup B(e)| \leq \Delta - 1$ and $|H(e)| \leq k - 1$. So we have that

$$\begin{aligned}
|X(e)| &= \left|X - \overline{X}(R(e))\right| \\
&\geq \Delta + k - 1 - |R(e)| \\
&= \Delta + k - 1 - |P(e) \cup B(e)| - |H(e)| \\
&\geq \Delta + k - 1 - (\Delta - 1) - (k - 1) \\
&= 1.
\end{aligned}$$

Thus, every edge has at least one eligible color. In particular, if $e \in U$, then $X(e) \neq \emptyset$. It will suffice to show that if e is uncolored, then at any time, Alice (or Bob for that matter) can color e with an eligible color such that both parts of the definition of legal color are satisfied. We will do this by proving two claims, one for each part of the definition of a legal color.

Claim 1. *Suppose $e \in U$. Then e has at most $2k$ children colored with colors from $X(e)$.*

Proof. Let e be an uncolored edge and let S be the subset of $S(e)$ composed of edges colored with colors from $X(e)$. Let $e' \in S$. Consider the time that e' is activated. Note by the definition of $X(e)$, no edge in $H[e]$ is colored with a color from $X(e)$. Thus, when e' is activated, Alice will not skip, as this would require that both e' and $f(e') \in H[e]$ are colored with a color in $X(e)$. Thus, Alice will take action in $H[e]$. So we have that $|S| \leq 2\,|H[e]| \leq 2k$. □

Claim 2. *Suppose $e \in U$ and $g \in S(e) \cap C$ with $c(g) \in X(e)$. Then $\operatorname{def}(g) \leq 2k^2 + 4k - 1$.*

Proof. Observe that since $B(g) \subset S(e)$ and $g \notin D(g)$, Claim 1 implies that $|D(g) \cap B(g)| \leq 2k - 1$. We also have that $P(g) = H[e]$, implying that

$P(g) \cap D(g) = \emptyset$. Finally, we note that $|D(g) \cap H(g)| \leq k-1$. Thus we have that

$$\begin{aligned} |D(g)| &= |D(g) \cap P(g)| + |D(g) \cap B(g)| + |D(g) \cap H(g)| + |D(g) \cap S(g)| \\ &\leq 3k - 2 + |D(g) \cap S(g)| . \end{aligned}$$

Thus, it will suffice to show that $|D(g) \cap S(g)| \leq 2k^2 + k + 1$.

Let $S = D(g) \cap S(g)$. We partition S into $\{S_1, S_2\}$. We define S_1 to be the set of edges e' where Alice responds to the activation of e' by jumping (and therefore taking action at an edge in $H[g]$). Similarly, S_2 is the set of edges e' where Alice responds to the activation of e' by skipping. At first, we have that $|S_1| \leq 2k$. However, this would imply that Alice both activates and colors the edges in $H[g]$. But she can color at most one edge in $H[g]$ with $c(g)$. Thus we have that $|S_1| \leq k+1$.

Now let $e' \in S_2$. Let $Q = \bigcup_{h \in H[g]} P(h)$. Since Alice will skip once e' is activated, she next will take action at an edge in Q. Therefore, as Alice can take action at most twice at any given edge, we have that $|S_2| \leq 2\,|Q| \leq 2k^2$.

Hence, we have

$$|S| = |S_1| + |S_2| \leq 2k^2 + k + 1,$$

as desired. □

Now suppose that Alice has chosen to color edge e with α. By Claim 1, this will result in $\mathrm{def}(e) \leq 2k$. If $g \in S(e)$ with $c(g) = \alpha$, then by Claim 2, when e is colored $\mathrm{def}(g) \leq 2k^2 + 4k - 1 + 1 = 2k^2 + 4k$. We make two final observations. First, Bob can always borrow this strategy to find a legal move. Second, if $d > 2k^2 + 4k$ and an edge e eventually has defect at least d then it must be through the actions of Bob that this has occurred. At the time an edge e is uncolored, the above arguments show that it is possible to color e with an eligible color α such that coloring e did not increase the defect of any edge e' where $\mathrm{def}(e') > 2k^2 + 4k$. Thus, the every edge will eventully be colored and Alice will win the game. ■

Since trees are 1-degenerate, Theorem 5 implies that $\mathrm{def}'_{\mathrm{g}}(T, \Delta) \leq 6$, but we can clearly do better using Theorem 3. This gives hope that there may be room for improvement in Theorem 5. We can apply Theorem 5 to outerplanar graphs and planar graphs, using the fact that outerplanar graphs are 2-degenerate and planar graphs are 5-degenerate.

Corollary 1. *Let G be a graph with $\Delta(G) = \Delta$.*

1. *If G is outerplanar, then $\mathrm{def}'_g(G, \Delta+1) \leq 16$. Moreover, if $d \geq 16$, then ${}^d\chi'_g(G) \leq \Delta + 1$.*
2. *If G is planar, then $\mathrm{def}'_g(G, \Delta+4) \leq 70$. Moreover, if $d \geq 70$, then ${}^d\chi'_g(G) \leq \Delta + 4$.*

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